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Characteristic Polynomial and Eigenvalues of Anti-adjacency Matrix for Graph $K_m \odot K_1$ and $H_m \odot K_1$

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ABSTRACT

Let $G = (V, E)$ be a simple and connected graph. The adjacency matrix G is a representation of a graph in the form of a square matrix, with the size of the matrix determined by the order G. By defining a graph into a matrix, lots of research related to a spectrum G has been done by researchers. Later, they defined the anti-adjacency matrix G as a matrix obtained by subtracting a matrix with all entries equal to one and the adjacency matrix G. In this paper, we determine the characteristic polynomial of matrix anti-adjacency for corona product between complete graph K_m and K_1 and hyper-octahedral graph H_m and K_1 with the eigenvalues.

Keywords: *Anti-adjacency matrix, Characteristic polynomial, Complete graph, Hyper-octahedral graph, Eigenvalue*

Introduction

The graph theory is one of the theories in mathematics. It was found by a mathematician named Euler who tried to solve a real-life problem related to the Konigsberg bridge. After the appearance of that theory in the eighteenth century, the graph theory began to develop and incited lots of theories and applications related to itself even with other branches in mathematics, for instance, algebra. Graph theory also helped to solve a problem related to transportation, optimization, and even modeling a molecular structure. In this paper, we deliver the definition of graph $G = (V, E)$ and the terminologies according to [1], which is defined as an ordered pair of (V, E) with $E \subset [V]^2$. The $[V]^2$ in here is defined as the collection of subset of V with cardinality two. Another terminology of graph we used here is related to adjacency. Two vertices in $G = (V, E)$, named u and v, are said adjacent if $uv \in E$. The vertex v and edge e is incident if $v \in e$. The complete graph with m vertices, denoted by $K_m, m \geq 1$, is a graph with a property that any two different vertices are adjacent. The graph K_1 also known as the trivial graph since it only has a vertex. Also, there is a hyper-octahedral graph, also known as cocktail party graph, denoted by $H_m, m \geq 1$, is a graph with $2m$ vertex such that the vertices can be labeled as $v_1, v_2, \dots v_{2m}$ and any two different vertices is adjacent except for $v_i v_{i+m}, i \in \{1, 2, \dots m\}$. The representation of graph in matrix has brought so many research related to it. It was started in the late 20th century with a focus on finding eigenvalues of graphs, also known as spectral theory. The spectral theory aims to find the eigenvalues together with the algebraic multiplicity. The spectra of some classes of graphs have been found, such as cycle graph, complete graph, hyper-octahedral graph, and others. Other than the adjacency matrix, research related to the incidence matrix and even the anti-adjacency matrix has been made worldwide. The incidence matrix of G is another representation of the graph with each row represented as the vertex of G, each column representing an edge of G, and the entry is zero or one depending on the incident relation between the vertex and edge. The adjacency matrix of G with order n, denoted by $\Gamma(G)$, is the square and symmetric matrix $n \times n$ with each row and column represent the vertex and each entry of the matrix is one the vertex that represents the row and column is adjacent and zero for otherwise [2].

The anti-adjacency matrix, denoted by $\Phi(G)$, is a subtraction of J and $\Gamma(G)$, where J is a square matrix with all entries equal to one [3]. Let us start defining the corona product between two graphs. Suppose we have G and H be two graphs. The corona product between G and H, denoted by $G \odot H$, is a graph obtained by taking a copy of G and $|V(G)|$ copies of G, then grafting i-th vertex of G and each vertex of i-th copy of H with an edge [4]. Wahri et al. [3] proved several results related to the characteristic polynomial of joined operation between two graphs. They also provided several results related to the characteristic polynomial of the fan graph, complete bipartite graph, and complement of a complete graph. Diwyacitta et al. [5] provided the characteristic polynomial of a directed cycle with chords. Edwina et al. [6] also provided the result related to the characteristic polynomial of disjoint union between two complete graphs, a wheel graph, and a star graph. Widiastuti et al. [7] determined the characteristic polynomial of a directed cyclic wheel graph. Following that, Aji et al. [8], Anzana et al. [9], and Hasyyati et al. [10] also provided the characteristic polynomial for the directed cyclic flower vase graph, friendship graph, and unicyclic corona graph respectively. Prayitno et al. [11] determine the characteristic polynomial of the line digraph. Based on these results, we determine the characteristic

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polynomial and the eigenvalues of the anti-adjacency matrix for graph corona product between complete graph and trivial graph, denoted by $K_m \odot K_1$, and graph corona product between hyper-octahedral graph and trivial graph, denoted by $H_m \odot K_1$.

Research Methods

The research method we use for this paper is literature research, where we gather all the necessary data, such as recent related research. We also gather the definitions and theorems from books related to graph theory, linear algebra, and matrix theory as the base of this research. After we gather that, we observe the theory used in the previous research, analyze it, and make use of the theory that supports this research. Finally, we construct the research results of this paper using the theory from previous research and books to determine the characteristic polynomial of $K_m \odot K_1$ and $H_m \odot K_1$together with the eigenvalues. Here is the theorem we used to prove the main results.

Theorem 1 [12] *If M is a matrix that can be partitioned as $M = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$ such that each partition is square matrix and $WY = YW$, then* $\det(M) = \det(WZ - XY)$.

Theorem 2. [13] *Suppose that A and B are two symmetric matrices and commute, then matrix A and B is simultaneously diagonalizable, that is there exist an orthogonal matrix P such that $A = PD_1P^T$ and $B = PD_2P^T$.*

Theorem 3. [14] *Suppose that A is square matrix of order n. If A is symmetric matrix, then there is an orthogonal matrix P, such that $A = PDP^T$ with D is a diagonal matrix consists of eigenvalues of A*

Theorem 4. [15] *Suppose that K_m is complete graph having order m and $\Gamma(K_m)$ as its adjacency matrix, then $spec\ K_m = \begin{pmatrix} m-1 & -1 \\ 1 & m-1 \end{pmatrix}$.*

Theorem 5. [2] *Suppose that H_m is a hyper-octahedral graph with order $2m$ and $\Gamma(H_m)$ as its adjacency matrix, then $spec\ H_m = \begin{pmatrix} 2m-2 & 0 & -2 \\ 1 & m & m-1 \end{pmatrix}$.*

Results and Discussion

The first theorem provided here is to state the characteristic polynomial and eigenvalues of graph corona product between complete graph K_m and trivial graph K_1.

Theorem 5. *Suppose that K_m is the complete graph with m vertices and $K_m \odot K_1$ is the corona product between graph K_m and trivial graph K_1, then $char(K_m \odot K_1) = (\lambda^2 - \lambda(m+1) - m^2 + 3m - 1)(\lambda^2 - \lambda - 1)^{m-1}$with eigenvalues $\lambda_1 = \frac{m(1+\sqrt{5})+1-\sqrt{5}}{2}, \lambda_2 = \frac{m(1-\sqrt{5})+1+\sqrt{5}}{2}, \lambda_3 = \frac{1+\sqrt{5}}{2}, \lambda_4 = \frac{1-\sqrt{5}}{2}$.*

Proof.
Let us first look into the adjacency matrix of $K_m \odot K_1$. The adjacency matrix of $K_m \odot K_1$, denoted by $\Gamma(K_m \odot K_1)$, is a matrix that can be partitioned into $\Gamma(K_m \odot K_1) = \begin{bmatrix} \Gamma(K_m) & I_m \\ I_m & 0_m \end{bmatrix}$ where $\Gamma(K_m)$ is an adjacency matrix of a complete graph. According to the definition of the anti-adjacency matrix, then

$$\Phi(K_m \odot K_1) = J_{2m} - \Gamma(K_m \odot K_1) = \begin{bmatrix} J_m - \Gamma(K_m) & J_m - I_m \\ J_m - I_m & J_m \end{bmatrix} \quad (1)$$

The characteristic polynomial of $K_m \odot K_1$, denoted by $\text{char}(\lambda) = \det\big(\lambda I_{2m} - \Phi(K_m \odot K_1)\big)$. By using Theorem 3, Theorem 4, and the definition of adjacency matrix, we get that

$$\Gamma(K_m) = PD^*P^T, D^* = diag\{m-1, -1, -1, \ldots, -1\} \quad (2)$$

Furthermore, since J is a matrix with all entries equal to one, then we can write $J = \Gamma(K_m) + I$. By this fact and according to Equation (2),

$$\begin{aligned} J &= PD^*P^T + PIP^T \\ &= P(D^* + I)P^T \\ &= P(diag\{m-1, -1, -1, \dots, -1\} + diag\{1, 1, 1, \dots, 1\})P^T \qquad (3) \\ &= P(diag\{m, 0, 0, 0, \dots, 0\})P^T \end{aligned}$$

Combining the Equation (1), (2), and (3) and using Theorem 1 since matrix $\left(\lambda_{I_m} - J_m + \Gamma(K_m)\right)$ and $(I_m - J_m)$ is commute, we get that

$$\begin{aligned} \text{char}(K_m \odot K_1) &= \det[\lambda I_{2m} - \Phi(K_m \odot K_1)] \\ &= \det \begin{bmatrix} \lambda I_m - J_m + \Gamma(K_m) & I_m - J_m \\ I_m - J_m & \lambda I_m - J_m \end{bmatrix} \\ &= \det\left[\left(\lambda I_m - J_m + \Gamma(K_m)\right)(\lambda I_m - J_m) - (I_m - J_m)^2\right] \\ &= \det[\lambda^2 I_m - 2\lambda J_m + mJ_m + \lambda\Gamma(K_m) - \Gamma(K_m)J_m - I_m + 2J_m - mJ_m] \\ &= \det\,[\lambda^2 I_m + \lambda(J_m - I_m) - 2\lambda J_m - (m-1)J_m - I_m + 2J_m] \\ &= \det\,[\lambda^2 I_m - \lambda J_m - \lambda I_m + 3J_m - mJ_m - I_m] \\ &= \det\,[P(\lambda^2 I_m - diag\{\lambda m + \lambda, \lambda, \lambda, \dots, \lambda\} - diag\{m(m-3), 0, 0, \dots, 0\} - I_m)P^T] \\ &= \det(P)\det\,(\,diag\{\lambda^2 - \lambda(m+1) - m^2 + 3m - 1, \lambda^2 - \lambda - 1, \dots, \lambda^2 - \lambda - 1\})\det\,(P^T) \\ &= \det(P)\,[\lambda^2 - \lambda(m+1) - m^2 + 3m - 1](\lambda^2 - \lambda - 1)^{m-1}\det(P^T) \\ &= \det(P)\det(P^T)\,[\lambda^2 - \lambda(m+1) - m^2 + 3m - 1](\lambda^2 - \lambda - 1)^{m-1} \\ &= (\lambda^2 - \lambda(m+1) - m^2 + 3m - 1)(\lambda^2 - \lambda - 1)^{m-1} \end{aligned}$$

As we get the characteristic polynomial, we can see that it contains two different polynomials at degree two. Therefore, to obtain the eigenvalues, we can only focus on finding the roots of these degree two polynomials. As a result, form the first polynomial, $\lambda^2 - \lambda(m+1) - m^2 + 3m - 1$, we get that the roots are $\lambda_1 = \frac{m(1+\sqrt{5})+1-\sqrt{5}}{2}, \lambda_2 = \frac{m(1-\sqrt{5})+1+\sqrt{5}}{2}$. Moreover, from the second polynomial, $\lambda^2 - \lambda - 1$, we get that the roots are $\lambda_3 = \frac{1+\sqrt{5}}{2}, \lambda_4 = \frac{1-\sqrt{5}}{2}$. So, these roots are the eigenvalues of the graph $K_m \odot K_1$.
■

For the last theorem, we also discover the characteristic polynomial and eigenvalues of graph corona product between hyper-octahedral graph H_m and trivial graph K_1.

Theorem 6. *Suppose that H_m is the hyper-octahedral graph with $2m$ vertices and $H_m \odot K_1$ is the corona product between H_m and trivial graph K_1, then $char(H_m \odot K_1) = \left(\lambda^2 - \lambda(2m+2) - (4m^2 - 8m + 1)\right)(\lambda^2 - 2\lambda - 1)^{m-1}(\lambda^2 - 1)^m$ with eigenvalues $\lambda_1 = \frac{2m+2+2\sqrt{5m^2-6m+2}}{2}$, $\lambda_2 = \frac{2m+2-2\sqrt{5m^2-6m+2}}{2}$, $\lambda_3 = \frac{2+2\sqrt{2}}{2}$, $\lambda_4 = \frac{2-2\sqrt{2}}{2}$, $\lambda_5 = 1$, $\lambda_6 = -1$.*

Proof.

The adjacency matrix of $H_m \odot K_1$ can be written as $\Gamma(H_m \odot K_1) = \begin{bmatrix} \Gamma(H_m) & I_{2m} \\ I_{2m} & 0_{2m} \end{bmatrix}$ with $\Gamma(H_m) = \begin{bmatrix} \Gamma(K_m) & J_m - I_m \\ J_m - I_m & \Gamma(K_m) \end{bmatrix}$. Since $\Gamma(H_m) \times J_{2m} = J_{2m} \times \Gamma(H_m) = (2m-2)J_{2m})$ and $x = [1, 1, 1, \dots, 1]^T$ is the eigenvector of eigenvalues $2m - 2$ in $\Gamma(H_m)$ and $2m$ in J_{2m}, therefore, according to Theorem 2

$$\begin{aligned} \Gamma(H_m) &= Pdiag\{2m-2, -2, -2, \dots, -2, 0, 0, \dots, 0\}P^T \\ \Gamma(J_{2m}) &= Pdiag\{2m, 0, 0, \dots, 0\}P^T \qquad (4) \end{aligned}$$

Moving to the anti-adjacency matrix of $H_m \odot K_1$, we get that

$$\Phi(H_m \odot K_1) = \begin{bmatrix} J_{2m} - \Gamma(H_m) & J_{2m} - I_{2m} \\ J_{2m} - I_{2m} & J_{2m} \end{bmatrix} \qquad (5)$$

Now, let us proceed to the characteristic polynomial of $H_m \odot K_1$. Note that matrix $[\lambda I - J + \Gamma(H_m)]$ and $(I_{2m} - J_{2m})$ commute, and based on the Equation (4), Equation (5) and Theorem 1, we have

$$
\begin{aligned}
&char(H_m \odot K_1) = \det[\lambda I_{2m} - \Phi(H_m \odot K_1)] \\
&= \det\begin{bmatrix} \lambda I_{2m} - J_{2m} + \Gamma(H_m) & I_{2m} - J_{2m} \\ I_{2m} - J_{2m} & \lambda I_{2m} - J_{2m} \end{bmatrix} \\
&= \det\left[\left(\lambda I_{2m} - J_{2m} + \Gamma(H_m)\right)(\lambda I_{2m} - J_{2m}) - (I_{2m} - J_{2m})^2\right] \\
&= \det\left[(\lambda I_{2m} - J_{2m})^2 + \lambda\Gamma(H_m) - \Gamma(H_m)J_{2m} - (I_{2m} - 2J_{2m} + 2mJ_{2m})\right] \\
&= \det[\lambda^2 I_{2m} - 2\lambda J_{2m} + 2mJ_{2m} + \lambda\Gamma(H_m) - (2\mathrm{m} - 2)J_{2m} - I_{2m} + 2J_{2m} - 2mJ_{2m}] \\
&= \det[\lambda^2 I_{2m} - 2\lambda J_{2m} + \lambda\Gamma(H_m) - (2\mathrm{m} - 2)J_{2m} - I_{2m} + 2J_{2m}] \\
&= \det[\lambda^2 I_{2m} - 2\lambda J_{2m} + \lambda\Gamma(H_m) - (2\mathrm{m} - 4)J_{2m} - I_{2m}] \\
&= \det[\lambda^2 I_{2m} - I_{2m} + \lambda. Pdiag\{2m-2, -2, -2, \dots, -2, 0, 0, \dots, 0\}P^T - 2\lambda. Pdiag\{2m, 0, 0, \dots, 0\}P^T \\
&\qquad -(2m-4). Pdiag\{2m, 0, 0, \dots, 0\}P^T] \\
&= \det[\lambda^2 I_{2m} - I_{2m} + Pdiag\{\lambda(2m-2) - 4\lambda m - (4m^2 - 8m), -2\lambda, -2\lambda, \dots, -2\lambda, 0, 0, \dots, 0\}P^T] \\
&= \det[\lambda^2 I_{2m} - I_{2m} + Pdiag\{\lambda(2m-2) - 4\lambda m - (4m^2 - 8m), -2\lambda, -2\lambda, \dots, -2\lambda, 0, 0, \dots, 0\}P^T] \\
&= \det[\lambda^2 I_{2m} - I_{2m} + Pdiag\{-2\lambda m - 2\lambda - (4m^2 - 8m), -2\lambda, -2\lambda, \dots, -2\lambda, 0, 0, \dots, 0\}P^T] \\
&= \det[Pdiag\{\lambda^2, \lambda^2, \dots, \lambda^2\}P^T - Pdiag\{1, 1, \dots, 1\}P^T \\
&\qquad -Pdiag\{2\lambda m + 2\lambda + (4m^2 - 8m), 2\lambda, 2\lambda, \dots, 2\lambda, 0, 0, \dots, 0\}P^T] \\
&= \det[P\, diag\{\lambda^2 - 1, \lambda^2 - 1, \dots, \lambda^2 - 1\}P^T - Pdiag\{2\lambda m + 2\lambda + (4m^2 - 8m), 2\lambda, \dots, 2\lambda, 0, \dots, 0\}P^T] \\
&= \det[Pdiag\{\lambda^2 - \lambda(2m+2) - (4m^2 - 8m + 1), \lambda^2 - 2\lambda - 1, \dots, \lambda^2 - 2\lambda - 1, \lambda^2 - 1, \dots \lambda^2 - \}P^T] \\
&= \det P\ \left(\lambda^2 - \lambda(2m+2) - (4m^2 - 8m + 1)\right)(\lambda^2 - 2\lambda - 1)^{m-1}(\lambda^2 - 1)^m \det P^T \\
&= \det P \det P^T \left(\lambda^2 - \lambda(2m+2) - (4m^2 - 8m + 1)\right)(\lambda^2 - 2\lambda - 1)^{m-1}(\lambda^2 - 1)^m \\
&= \left(\lambda^2 - \lambda(2m+2) - (4m^2 - 8m + 1)\right)(\lambda^2 - 2\lambda - 1)^{m-1}(\lambda^2 - 1)^m
\end{aligned}
$$

Therefore, since the characteristic polynomial is delivered as multiplication of degree two polynomials, then the eigenvalues are the roots of these polynomials. For the first polynomial, $\lambda^2 - \lambda(2m+2) - (4m^2 - 8m + 1)$, the roots are $\lambda_1 = \frac{2m+2+2\sqrt{5m^2-6m+2}}{2}$, $\lambda_2 = \frac{2m+2-2\sqrt{5m^2-6m+2}}{2}$. For the polynomial inside square roots, it is routine to see that for $m \geq 1$, the value is positive. The second polynomial is $\lambda^2 - 2\lambda - 1$ and the roots are $\lambda_3 = \frac{2+2\sqrt{2}}{2}$, $\lambda_4 = \frac{2-2\sqrt{2}}{2}$. And lastly, the polynomial is $\lambda^2 - 1$ and the roots are $\lambda_5 = 1$, $\lambda_6 = -1$. Finally, we get the six eigenvalues that are the roots of three polynomials.
■

Conclusion

The anti-adjacency matrix of graph corona product between complete graph K_m and K_1 can be represented as a block matrix. Moreover, the characteristic polynomial of $K_m \odot K_1$ is $(\lambda^2 - \lambda(m+1) - m^2 + 3m - 1)(\lambda^2 - \lambda - 1)^{m-1}$ with eigenvalues $\lambda_1 = \frac{m(1+\sqrt{5})+1-\sqrt{5}}{2}, \lambda_2 = \frac{m(1-\sqrt{5})+1+\sqrt{5}}{2}, \lambda_3 = \frac{1+\sqrt{5}}{2}, \lambda_4 = \frac{1-\sqrt{5}}{2}$. Besides that, the characteristic polynomial of the graph obtained by the corona product between the hyper-octahedral graph H_m and trivial graph K_1 is $\left(\lambda^2 - \lambda(2m+2) - (4m^2 - 8m + 1)\right)(\lambda^2 - 2\lambda - 1)^{m-1}(\lambda^2 - 1)^m$ with eigenvalues $\lambda_1 = \frac{2m+2+2\sqrt{5m^2-6m+2}}{2}, \lambda_2 = \frac{2m+2-2\sqrt{5m^2-6m+2}}{2}, \lambda_3 = \frac{2+2\sqrt{2}}{2}, \lambda_4 = \frac{2-2\sqrt{2}}{2}, \lambda_5 = 1, \lambda_6 = -1$. For further research, we can determine the characteristic polynomial and the eigenvalues of graph $G \odot K_1$ with any circulant graph G, even for graph $G \odot \overline{K_n}$.

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