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Dynamic Analysis of a Prey Predator Model with Holling-Type III Functional Response and Anti-Predator Behavior

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ABSTRACT

The prey-predator model in this article reviews the interaction of two populations with a type III Holling -type III functional response and anti-predator behavior. Dynamic analysis starts by determining the basic model construction assumptions, equilibrium and stability points, and numerical simulations using Python. Dynamic analysis results obtained four equilibrium points with types of stability, namely $\boldsymbol{E_0(0,0)}$ *which is unstable,* $\boldsymbol{E_1(k,0)}$ *which is asymptotically stable* $\boldsymbol{E_2}$ *, and* $\boldsymbol{E_3}$*. Which is stable under certain conditions. The numerical simulation results show double stability at the equilibrium points.* $\boldsymbol{E_1}$ *and* $\boldsymbol{E_2}$ *with the anti-predator behavior parameter value* $\boldsymbol{\eta = 0.01}$*. The anti-predator parameter value* $\boldsymbol{\eta = 0.473}$ *indicates a change in stability that is only the* $\boldsymbol{E_1}$ *equilibrium point. Differences in the values of the anti-predator behavior parameters affect changes in system solutions and impact reducing predator populations.*

Keywords: *Anti-Predator Behaviour, Holling Type III, Dynamics Analysis, Bistable System*

Introduction

All populations on Earth cannot live alone but need and interact with each other. The interaction process includes the process of eating and being eaten by one animal species with another animal. Animals that eat other animals are called predators, while animals that other animals hunt are called prey [1]. The purpose of interaction between predators and their prey is to control the density of prey populations to maintain their lives [2], [3].

Alfred Lotka and Vito Volterra introduced the prey-predator interaction model, namely the Lotka-Voltera model [4]. In 1953, Holling introduced an expanded Lotka-Voltera model of predation called the response function Holling proposed four types of response functions: type I, type II, type III, and type IV response functions [5], [6].

In general, the interaction process between prey and predators is assumed to be that the prey group is always caught and eaten by predators [7], [8]. Not all prey can be captured and eaten by predators, but some can attack or eat other individuals. In this case, some prey could fight and escape the predator. Such is the case with the Red Colobus Monkey (*Piliocolobus)*, which protects itself by shouting alarm calls for its group when threatened by Chimpanzees [9]. Therefore, pressure from predators that causes prey to escape and fight back from predators is known as anti-predator behavior [10].

Researchers developed the Lotka Volterra prey-predator model, which involves anti-predator behavior. Until now, few studies have discussed the dynamic analysis of the prey-predator system with the Type III Holling response function and anti-predator behavior. This research refers to two studies, including [11] studied a predator-prey model with a Holling Type IV response function and anti-predator behavior. Supported by other studies,[12]–[14] examines the interaction model of prey-predator with ratio-dependent function and anti-predator behavior. The complex dynamics of the model with the phenomenon of anti-predator behavior in nature have also been discussed [15]–[17].

Inspired by these two studies, researchers still consider anti-predator behavior by modifying it using the Holling type III functional response. Holling type III active response is suitable because it has the problem of predators that tend to look for other prey populations when the prey population continues to decrease due to self-protection, namely anti-predator behavior.

Research Method

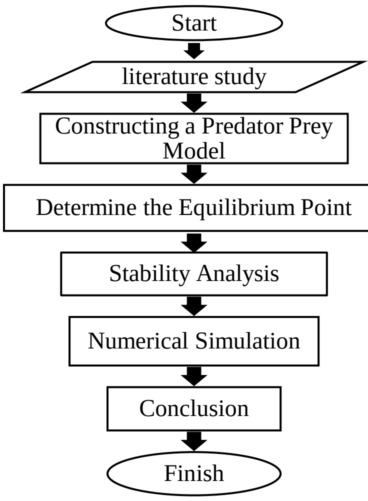

Figure 1.Research Flow Chart

Based on Figure 1, the stages begin with a literature study, constructing a model related to assumptions, dynamic analysis, which includes equilibrium points, determining the type of stability, determining parameters for numerical simulations with a Python program illustrated with phase portraits, and drawing conclusions based on the results of the analysis.

Result and Discussion

Construction of a Prey-Predator Model

Based on the same assumption in [6], [11], the prey-predator interaction model obtained uses a different functional response, namely Holling type III and anti-predator behavior. The growth rate of the prey population follows the logistic equation. Construction is denoted.

$$\frac{dx}{dt} = \alpha x\left(1 - \frac{x}{K}\right) - \frac{bx^2y}{n^2+x^2},$$
$$\frac{dy}{dt} = \frac{\delta bx^2y}{n^2+x^2} - cy - \eta xy. \quad (1)$$

The dynamical system (1) has been analysed in the region $\{(x, y) \in \mathbb{R}_+^2 : x \geq 0, y \geq 0\}$ with the initial conditions $x(0) > 0, y(0) > 0$. Here, $x(t)$ dan $y(t)$ denote the numbers of prey and predator. The predators consume the prey with Holling type III functional response $\frac{x^2y}{n^2+x^2}$ for the capture rate $b,$ and continue their growth with conversion rate δ. Parameter K represents environmental carrying capacity, α is the maximum per capita prey growth rate, c is a mortality rate of a predator in the absence of prey, n namely the level of predator saturation, and η anti-predator behavior.

Dynamic Analysis of Prey-Predator Models

a. The Equilibrium Point

The equilibrium point in a model (1) is obtained by solving the system of equations that makes $\frac{dx}{dt} = 0$ dan $\frac{dy}{dt} = 0$ [18]. System (1) has different points of equilibrium. E_0, E_1, and E_2.

1. $E_0(0,0)$ declared the extinction of prey and predator populations.
2. $E_1(k, 0)$ declared the predator population to be extinct.
3. $E_2(x^*, y^*)$ the interior equilibrium point that both predator and prey populations exist.

Let's say $x = x^*$ with x as the value of the equilibrium point x^* which is obtained by simplifying the equation (1) so the value of y^* must be positive as follows.

$$y^* = -\frac{a}{bkx^*}\left(kn^2 + n^2x^* + kx^{*2} + kx^{*3}\right). \tag{2}$$

Considering the presence of positive roots from equation (2) according to Formula Cardano [19], [20]. The value x^* is the positive solution of the following cubic equation.

$$x^{*3} + 3\omega_1 x^{*2} + 3\omega_2 x^* + \omega_3 = 0. \tag{3}$$

Here is the value of each component.

$$\omega_1 = \frac{c-\delta b}{3\eta},$$
$$\omega_2 = \frac{n^2}{3},$$
$$\omega_3 = \frac{cn^2}{\eta}.$$

Hence, system (1) has a unique positive equilibrium. $E_2{}^*(x^*, y^*)$.

b. Local Stability of Equilibrium

Stability analysis is determined by the linearization process on the system arranged in the form of a matrix [21]. The linearization results use the following Jacobian matrix:

$$J(x,y) = \begin{bmatrix} \alpha - \frac{2\alpha x}{k} - \frac{2bxy}{n^2+x^2} + \frac{2bx^3y}{(n^2+x^2)^2} & -\frac{bx^2}{n^2+x^2} \\ \frac{2\delta bxy}{n^2+x^2} - \frac{2\delta bx^3y}{(n^2+x^2)^2} - \eta y & \frac{\delta bx^2}{n^2+x^2} - c - \eta x \end{bmatrix}. \tag{4}$$

We obtain the following theorem of the stability of the system's equilibrium (1).

Theorem 1. Equilibrium point $E_0(0,0)$ is unstable (saddle point)

Proof. The Jacobian matrix in equation (4) at $E_0(0,0)$ is as follows:

$$J_{E_0} = \begin{bmatrix} \alpha & 0 \\ 0 & -c \end{bmatrix}. \tag{5}$$

The eigenvalues Jacobian matrix (5) is.

$$\lambda_1 = \alpha, \text{ and } \lambda_2 = -c.$$

It's evident that $\alpha > 0$, so as $\lambda_1 > 0$ and $c < 0$ so as $\lambda_2 < 0$. Therefore, E_0 is unstable (saddle point)[22].

Theorem 2. Equilibrium point $E_1(k, 0)$ is asymptotically stable (node) if the following conditions are satisfied $\frac{\delta bk^2}{n^2+k^2} < c + \eta k$.

Proof. At the point $E_1(K, 0)$, the Jacobian matrix in equation (4) becomes

$$J_{E_1} = \begin{bmatrix} -\alpha & -\frac{bk^2}{n^2+x^2} \\ 0 & \frac{\delta bk^2}{n^2+k^2} - c - \eta k \end{bmatrix}. \tag{6}$$

The eigenvalues of the Jacobian matrix J_{E_1} (6) are.

$$\lambda_1 = -\alpha, \text{ dan } \lambda_2 = \frac{\delta b k^2}{n^2+k^2} - c - \eta k.$$

Therefore, both eigenvalues of the Jacobian matrix J_{E_1} have negative genuine parts $\alpha < 0$ so as $\lambda_1 < 0$, and to conditions is satisfied if $\lambda_2 < 0$

$$\frac{\delta b k^2}{n^2+k^2} < c + \eta k.$$

in that $\lambda_1 < 0$ and $\lambda_2 < 0$, then the equilibrium point E_1 is asymptotically stable (node) with conditions $\frac{\delta b k^2}{n^2+k^2} < c + \eta k.$

Theorem 3. Equilibrium point $E_2(x^*, y^*)$ is asymptotically stable (node) if the condition:

(i) $(m_{11} + m_{22}) < 0,$

(ii) $m_{11}{}^2 + m_{22}{}^2 + 4m_{12}m_{21} > 2m_{11}m_{22}.$

Proof. We evaluate the Jacobian matrix at $E_2(x^*, y^*)$ to obtained:

$$J_{E_2} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}.$$

The respective components are given by:

$$m_{11} = \alpha - \frac{2\alpha x^*}{k} - \frac{2bx^*y^*}{n^2+x^{*2}} + \frac{2bx^{*3}y^*}{(n^2+x^{*2})^2},$$

$$m_{12} = -\frac{bx^{*2}}{n^2+x^{*2}},$$

$$m_{21} = \frac{2\delta bx^*y^*}{n^2+x^{*2}} - \frac{2\delta bx^3y^*}{(n^2+x^{*2})^2} - \eta y^*,$$

$$m_{22} = \frac{\delta bx^{*2}}{n^2+x^{*2}} - c - \eta x^*.$$

The characteristic equation of J_{E_2} is given by:

$$\lambda^2 + \tau\lambda + \sigma = 0, \qquad (7)$$

From characteristic equation (7), the eigenvalue of J_{E_2} are given by:

$$\lambda_{1,2} = \frac{(\tau) \pm \sqrt{(\tau)^2 - 4(\sigma)}}{2},$$

if $\tau = (m_{11} + m_{22}) < 0$ and $\sigma = (m_{11} + m_{22})^2 - 4(m_{11}m_{22} - m_{12}m_{21}) > 0$, then the equilibrium point E_2 is asymptotically stable with conditions $(m_{11} + m_{22}) < 0$, and $m_{11}{}^2 + m_{22}{}^2 + 4m_{12}m_{21} > 2m_{11}m_{22}$.

c. Numerical Simulation

Numerical simulations show dynamic behavior changes around the equilibrium point [23]. To perform a numerical simulation, the values of the following parameters are required.

Table 1. System Parameter Value.

Parameter	Value	Reference
α	6.8	Assumptions
k	7.5	Assumptions
δ	0.31	Assumptions
c	0.015	Tang & Xiao [11]
b	0.4	Tang & Xiao [11]
n	0.1	Ayah [24]
η	0.1	Tang & Xiao [11]

Based on parameters in Table 1, with $\eta = 0.1$ four equilibrium points exist from the eigenvalue results to know the type of stability.

Table 2. Parameter Stability

Equilibrium Points	Eigen Values	Stability Type
$E_0(0,0)$	$\lambda_1 = -0.015 < 0, \quad \lambda_2 = 6.8 > 0$	Unstable (saddle point)
$E_1(7.5,0)$	$\lambda_1 = -6.8 < 0, \quad \lambda_2 = -0.641 < 0$	Asymptotic Stability (node)
$E_2(0.042, 4.66)$	$\lambda_1 = -4.65 < 0, \quad \lambda_2 = -0.04 < 0$	Asymptotic Stability (node)
$E_3(1.079, 15.84)$	$\lambda_1 = 4.87 > 0, \quad \lambda_2 = -0.126 < 0$	Unstable (saddle point)

In Table 3, two equilibrium points are obtained that meet the conditions for the existence of stability, namely, the equilibrium point at E_1 and equilibrium point E_2. Therefore, a system with stability at two equilibrium points is called bistable. Stability changes of system behavior in equations (1) and (2) are shown through numerical simulations by increasing the value of the anti-predator behavior parameter the first $\eta = 0.1$ with $\eta = 0.473$. The initial values used for the numerical simulations are the same as those [17,17] and [1, 17], which produce phase portraits.

Numerical simulation results based on the parameter values in Table 2 with different parameter values on anti-predator behavior $\eta = 0.1$ illustrated in Figure 2 and Figure 3.

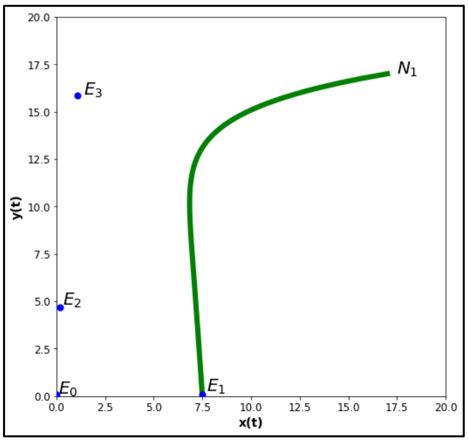

Figure 2. Phase Portrait Convergent to E_1

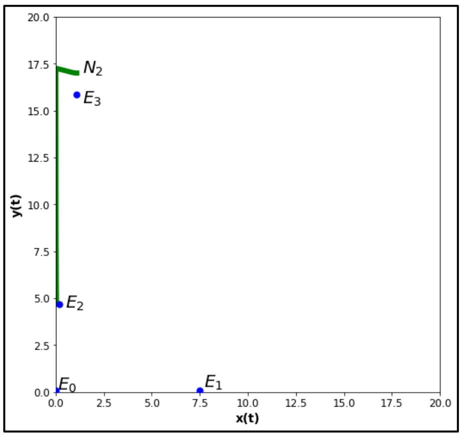

Figure 3. Phase Portrait Convergent to E_2

The system (1) can exhibit bistability at η = 0.1. In Figure (2) and Figure (3), the system has double stability, namely bistable at E_1 and E_2 influenced by differences in initial values. Figure (2) shows the solution from a convergent system to an equilibrium point $E_1(7.5,0)$ using initial values $N_1[17,17]$ so that the equilibrium point E_1 is stable. Figure 3 shows the solution from a convergent system to an equilibrium point $E_3(0.04, 4.66)$ using initial values $N_2[1,17]$ so that the equilibrium point E_2 is stable.

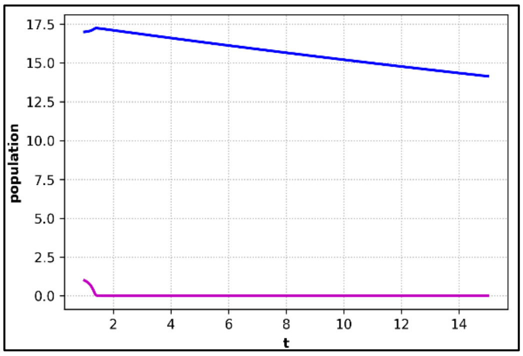

Figure 4. Stable Time Series Graphs at E_1 with Initial Values [17, 17]

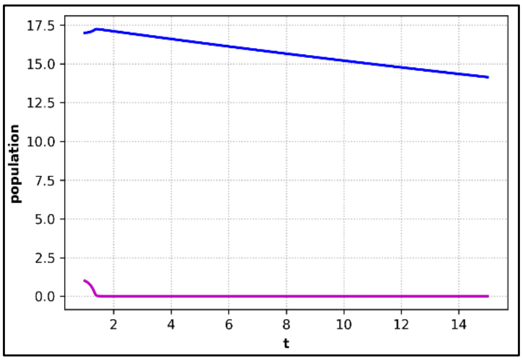

Figure 5. Stable Time Series Graphs at E_2 with Initial Values [1, 17]

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Numerical simulation with increasing parameter values on anti-predator behavior is $\eta = 0.473$ as illustrated in the figure below:

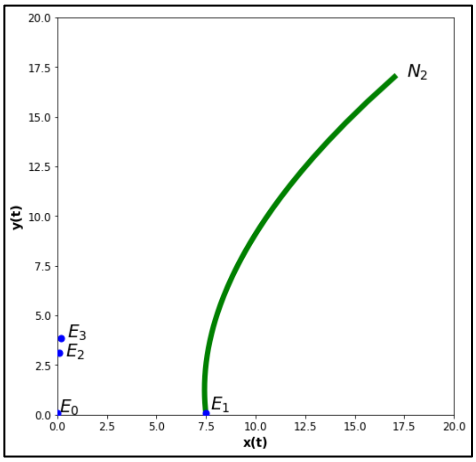

Figure 6. Phase Potrait Convergent to E_1

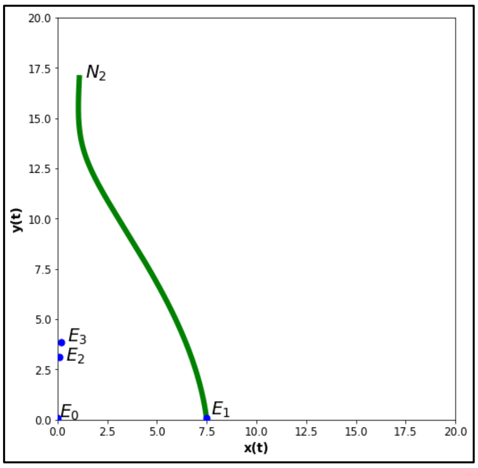

Figure 7. Phase Portrait Convergent to E_2

Figure 6 and Figure 7 show that parameter anti-predator behavior is $\eta = 0.473$ produce system has stability only at E_1 with the difference in the initial value $N_1[17,17]$ and $N_2[1, 17]$. Both show a solution from a convergent system to an equilibrium point. $E_1(7.5,0)$ so that the equilibrium point E_1 are stable.

The results of the two simulations show that anti-predator behavior can affect a change in the system's stability. When the parameter is anti-predator behavior $\eta = 0.1$ results in a bistable system that is at the equilibrium point E_1 which states the population of predators experiencing extinction and the equilibrium point E_2 states that both populations exist. The appearance of a bistable system indicates that the system solution is influenced by the initial values of the two people. When the parameter of anti-predator behavior increases to $\eta = 0.473$ it produces stability, namely at the equilibrium point E_1 the predator population is experiencing extinction because they cannot live without predation.

Conclusion

Construction of a prey-predator model with a Holling type III response function and anti-predator behavior is.

$$\frac{dx}{dt} = \alpha x\left(1 - \frac{x}{K}\right) - \frac{bx^2y}{n^2 + x^2},$$
$$\frac{dy}{dt} = \frac{\delta bx^2y}{n^2+x^2} - cy - \eta xy.$$

The results of the Dynamic Analysis produce three equilibrium points, namely. $E_0(0,0)$ declaring extinction in two populations, $E_1(k, 0)$ declared prey populations to extinction, and E_2 as well as E_3 states that both populations exist. Stability analysis of the equilibrium points with the type of stability of each, namely E_0 unstable. Except for $E_1, E_2,$ and E_3 stable under certain conditions of existence. Differences in the parameter values of anti-predator behavior are carried out to determine the effect of anti-predator behavior on changes in system solutions. The numerical simulation results show a bistable system when the parameter anti-predator behaviour is $\eta = 0.1$, which means that both prey populations can coexist and the predator populations experience extinction. When the parameter anti-predator behaviour is $\eta = 0.473$ to produce a stable system, then it means that the predator population has experienced demise.

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