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New Opportunity Model Using a Mix of 7 and 8 Gamma Chance Density Functions

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Abstract - *This research is based on the limitations of classic distributions such as the Normal, Exponential, and Weibull in modeling real-world data with skewness, heavy tails, multimodality, or hazard rate structures, and thus it is necessary to develop a new, more flexible opportunity model. The main objective of this study is to develop a new probability model with a single parameter using a mixed-method approach that specifically employs the 7- and 8 Gamma chance density functions. The research methodology includes the reduction of important characteristics such as the cumulative distribution function, survival function, and hazard function, as well as the use of the Maximum Likelihood method with Newton–Raphson numerical iteration for parameter estimation. The results of this study are expected to prove that the single-parameter 7 and 8 Gamma mixed models have superior performance compared to the previously existing single-parameter models. The contributions of this research include theoretical development in statistical science and the provision of efficient, parsimonious alternative models for application in fields such as biostatistics, finance, and reliability analysis.*

Keywords: *Gamma distribution mixtures, statistical characteristics, maximum method of likelihood. New opportunity model, one parameter.*

1. Introduction

The development of new opportunity models has become a central theme in statistical research. Classic distributions such as Normal, Exponential, or Weibull often lack sufficient flexibility when modeling real data that exhibit skewness, heavy tails, multimodality, or complex hazard rate structures. These limitations have prompted statisticians to introduce new distribution families with additional parameters and transformation mechanisms to improve model flexibility. For example, the Odd Exponentiated Half-Logistic distribution emphasizes that traditional models are often insufficient to capture the variations observed in real data [1]. Similarly, an extended family of distributions argues that adding parameters is essential to improve adaptability across various applications [2]. A new generalized distribution family to meet challenges in biostatistics and reliability analysis [3]. In line with this, the Odd Chen-G family of distributions emphasizes the need for models capable of describing complex hazard-rate behavior [4]. More recently, the Generalized Transmuted-G distribution family [5] further reinforces the idea that the continuous emergence of new opportunity models is critical to modern data analysis.

Overall, this cutting-edge contribution suggests that the proliferation of new opportunities is not just an academic trend but a statistical necessity. The new distribution provides more reliable tools for more accurate modeling, inference, and prediction in areas such as survival analysis, engineering reliability, finance, and environmental science.

In the development of opportunity models, adding parameters often increases the flexibility of distributions, enabling them to describe different forms of data. However, both classical and modern statistical literature confirm that models with too many parameters are not always better. Excessive complexity can create estimation problems, increasing the risk of reducing model interpretability.

The principle of parsimony, which is that a model with a smaller number of parameters is preferred as long as it is still able to explain the data adequately [6]. Correspondingly, simple models often provide the best balance between accuracy and complexity [7]. Simplicity is a key factor in model selection, as complex models do not always achieve better predictive performance than simpler ones [8].

In addition, information-based model selection criteria, such as AIC, tend to favor models with fewer parameters if the fit performance is relatively comparable [9]. Practical applications: simple models are easier to interpret and more predictive and robust than models with many parameters [10].

Thus, although the emergence of new opportunity distributions with additional parameters is essential to expand model flexibility, the current literature still indicates that models with fewer parameters are often superior in terms of efficiency, interpretability, and generalization

2. Research Methods

2.1. *Types of Research*

The research used to complete this final project will produce a new opportunity model with 1 parameter. Data from previous researchers will be used to compare the new opportunity model with the existing one. The benefits of opportunity models will be examined to determine the advantages of the new models.

2.2. Research Steps

There are several stages used to obtain the level of population density, including:

a. Collecting probability models with 1 parameter that have been generated by previous researchers
b. Analysis of the Development of Opportunity Models 1 Parameter
c. Determine general patterns of change in opportunity models
d. Generating new opportunity models from known common patterns
e. Generate important characteristics of new opportunity models
f. Estimation of parameters of the new opportunity model
g. Test the Benefits of New Opportunity Models That Have Been Produced
h. Compare the results of the analysis of the new opportunity model against the existing opportunity model by applying it to specific data
i. Conclusion.

3. Discussion

3.1 Mixed Opportunity Density Function 7 Gamma Distribution and 8 Gamma Distribution

The mixture *distribution function* has a very important role in producing a new opportunity model or opportunity density function. The definition of the mixed opportunity density function has been discussed in the previous section, from this definition it can be ascertained that the mixed opportunity density function meets the properties of an opportunity density function among which are . Mixed chance density functions resulting from multiple mixtures of Gamma chance density functions (α,θ) for positive integer α with specific weights have resulted in several new chance density functions. Gamma (α,θ) can be defined as . The function of the density of these new opportunities will be discussed in this section.$f(y)\int_{-\infty}^{\infty} f(y)\,dy = 1, -\infty \le y \le \infty f(y) = \frac{\theta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-y\theta}, y, \theta > 0 \; dan \; \Gamma(\alpha) = (\alpha-1)!$. untuk α bilangan bulat positif.

3.2 Model – Opportunity Model 1 Parameter

Based on previous research as stated in Table 4.1, it is concluded that new opportunity models for 1 parameter can continue to develop into new opportunity models by increasing the number of gamma opportunity density function mixes with certain parameters and ballasts.

Table 1 Research Data

Distribution	***f (x)***
Lindley	$\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}$
Sujatha	$\frac{\theta^3}{\theta^2+\theta+2}(1+x+x^2)e^{-\theta x}$
Amarendra	$\frac{\theta^4}{\theta^3+\theta^2+2\theta+6}(1+x+x^2+x^3)e^{-\theta x}$
Düsseldorf	$\frac{\theta^5}{\theta^4+\theta^3+2\theta^2+6\theta+24}(1+x+x^2+x^3+x^4)e^{-\theta x}$
Shambhu	$\frac{\theta^6}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120}(1+x+x^2+x^3+x^4+x^5)e^{-\theta x}$

3.3 Common Patterns Towards Model Change – Opportunity Models

The results of the analysis of the development of one-parameter opportunity models produced through the Gamma distribution mix technique, ranging from Lindley distribution to Shambu distribution, obtained a general pattern of opportunity model development as follows:

$$f(y) = \frac{\theta^k}{\sum_{i=1}^{k} \Gamma(i)\theta^{k-1}}(1 + \sum_{i=1}^{k-1} x^i)e^{-\theta x}$$

3.4 New Opportunity Model Results from Known Common Patterns

a. Mixed Opportunity Density Function 7 Gamma Distribution

The process of forming a mixed density function of 7 Gamma Distribution can be done by understanding the technique of the mixed opportunity density function that has been discussed earlier, the process can be explained as follows.

1) Set Gamma (1,θ) and weights w_1
2) Set Gamma (2,θ) and weights w_2
3) Set Gamma (3,θ) and ballast w_3
4) Set Gamma (4,θ) and ballast w_4
5) Set Gamma (5,θ) and ballast w_5
6) Set Gamma (6.θ) and ballast w_6
7) Set Gamma (7,θ) and weights w_7
8) Make sure that $\sum_{i=1}^{7} w_i = 1$

The Opportunity Density function with each weighter is given in table 2:

Table 3.2 Gamma Density and Weighting Functions For Mixtures of 7 Gamma Distributions

Distribution	Opportunity Density Function/f(y)	Ballast
Gamma (1.θ)	$f_1(y) = \theta e^{-y\theta}$	$w_1 = \frac{\theta^6}{\theta^6 + \theta^5 + 2\theta^4 + 6\theta^3 + 24\theta^2 + 120\theta + 720}$
Gamma (2.θ)	$f_2(y) = \theta^2 y e^{-y\theta}$	$w_2 = \frac{\theta^5}{\theta^6 + \theta^5 + 2\theta^4 + 6\theta^3 + 24\theta^2 + 120\theta + 720}$
Gamma (3.θ)	$f_3(y) = \frac{\theta^3}{2} y^2 e^{-y\theta}$	$w_3 = \frac{2\theta^4}{\theta^6 + \theta^5 + 2\theta^4 + 6\theta^3 + 24\theta^2 + 120\theta + 720}$
Gamma (4.θ)	$f_4(y) = \frac{\theta^4}{6} y^3 e^{-y\theta}$	$w_4 = \frac{6\theta^3}{\theta^6 + \theta^5 + 2\theta^4 + 6\theta^3 + 24\theta^2 + 120\theta + 720}$
Gamma (5.θ)	$f_5(y) = \frac{\theta^5}{24} y^4 e^{-y\theta}$	$w_5 = \frac{24\theta^2}{\theta^6 + \theta^5 + 2\theta^4 + 6\theta^3 + 24\theta^2 + 120\theta + 720}$
Gamma (6.θ)	$f_6(y) = \frac{\theta^6}{120} y^5 e^{-y\theta}$	$w_6 = \frac{120\theta}{\theta^6 + \theta^5 + 2\theta^4 + 6\theta^3 + 24\theta^2 + 120\theta + 720}$
Gamma (7,θ)	$f_7(y) = \frac{\theta^7}{720} y^6 e^{-y\theta}$	$w_7 = \frac{720}{\theta^6 + \theta^5 + 2\theta^4 + 6\theta^3 + 24\theta^2 + 120\theta + 720}$

b. Mixed Opportunity Density Function 8 Gamma Distribution

The process of forming the mixed density function of 8 Gamma Distribution can be done in the same way as the previous 7 Gamma distribution mixtures. This density function is carried out by adding a mixture with Gamma (8, θ), so that the process of forming the mixed density function of 8 Gamma distribution can be explained as follows:

1) Set Gamma (1,θ) and weights w_1
2) Set Gamma (2,θ) and weights w_2
3) Set Gamma (3,θ) and ballast w_3
4) Set Gamma (4,θ) and ballast w_4
5) Set Gamma (5,θ) and ballast w_5
6) Set Gamma (6.θ) and ballast w_6
7) Set Gamma (7,θ) and weights w_7
8) Set Gamma (8,θ) and ballast w_8
9) Make sure that $\sum_{i=1}^{8} w_i = 1$

The Opportunity Density function and weights required to produce the mixed chance density function of 8 Gamma distributions are given in Table 3.

Table 3.3 Gamma Density and Ballast Functions For 8 Gamma Distribution Mixtures

Distribution	Opportunity Density Function/f(y)	Ballast
Gamma (1.θ)	$f_1(y) = \theta e^{-y\theta}$	$w_1 = \frac{\theta^7}{\theta^7+\theta^6+2\theta^5+6\theta^4+24\theta^3+120\theta^2+720\theta+5040}$
Gamma (2.θ)	$f_2(y) = \theta^2 y e^{-y\theta}$	$w_2 = \frac{\theta^6}{\theta^7+\theta^6+2\theta^5+6\theta^4+24\theta^3+120\theta^2+720\theta+5040}$
Gamma (3.θ)	$f_3(y) = \frac{\theta^3}{2} y^2 e^{-y\theta}$	$w_3 = \frac{2\theta^5}{\theta^7+\theta^6+2\theta^5+6\theta^4+24\theta^3+120\theta^2+720\theta+5040}$
Gamma (4.θ)	$f_4(y) = \frac{\theta^4}{6} y^3 e^{-y\theta}$	$w_4 = \frac{6\theta^4}{\theta^7+\theta^6+2\theta^5+6\theta^4+24\theta^3+120\theta^2+720\theta+5040}$
Gamma (5.θ)	$f_5(y) = \frac{\theta^5}{24} y^4 e^{-y\theta}$	$w_5 = \frac{24\theta^3}{\theta^7+\theta^6+2\theta^5+6\theta^4+24\theta^3+120\theta^2+720\theta+5040}$
Gamma (6.θ)	$f_6(y) = \frac{\theta^6}{120} y^5 e^{-y\theta}$	$w_6 = \frac{120\theta^2}{\theta^7+\theta^6+2\theta^5+6\theta^4+24\theta^3+120\theta^2+720\theta+5040}$
Gamma (7,θ)	$f_7(y) = \frac{\theta^7}{720} y^6 e^{-y\theta}$	$w_7 = \frac{720\theta}{\theta^7+\theta^6+2\theta^5+6\theta^4+24\theta^3+120\theta^2+720\theta+5040}$
Gamma (8,θ)	$f_8(y) = \frac{\theta^8}{5040} y^7 e^{-y\theta}$	$w_8 = \frac{5040}{\theta^7+\theta^6+2\theta^5+6\theta^4+24\theta^3+120\theta^2+720\theta+5040}$

3.5 Important Characteristic Result of New Opportunity Models

a. Mixed Opportunity Density Function 7 Gamma Distribution

The mixed chance density function of 7 Gamma distribution can be generated by using the concept of the mixed chance density function (2.11) as follows:

$$f(y) = \sum_{i=1}^{7} w_i\, f_i(y) = w_1 f_1(y) + w_2 f_2(y) + \cdots + w_7 f_7(y)$$

$$= \frac{\theta^6}{\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720}\theta e^{-y\theta} + \cdots + \frac{720}{\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720}\frac{\theta^7}{720} y^6 e^{-y\theta}$$

$$= \frac{\theta^7}{\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720}(1+y+y^2+\cdots+y^6)e^{-y\theta}$$

$$= \frac{\theta^7}{\Gamma(1)\theta^6+\Gamma(2)\theta^5+\Gamma(3)\theta^4+\Gamma(4)\theta^3+\Gamma(5)\theta^2+\Gamma(6)\theta+\Gamma(7)}(1+y+\cdots+y^6)e^{-y\theta}$$

$$= \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\left(\sum_{i=0}^{7-i} y^i\right)e^{-y\theta}$$

The mixed probability density function of the 7 Gamma distribution above can be said to be true if it can meet the nature of an opportunity density function, which is to prove (2.1). This proof relies heavily on the integral properties of the gamma chance density function as follows (2.10). So that the proof of these properties can be done as follows: $\int_{-\infty}^{\infty} f(y)\,dy = 1$ $\int_0^{\infty} y^{\alpha-1}e^{-y\theta}\,dy = \frac{\Gamma(\alpha)}{\theta^\alpha}$

$$\int_0^{\infty}\frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{k-i}}\sum_{i=0}^{7-1} y^i\, e^{-y\theta}\,dy = \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{k-i}}\int_0^{\infty}(1+y+\cdots+y^{7-1})\,e^{-y\theta}dy$$

$$= \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\left(\int_0^{\infty} e^{-y\theta}\,dy + \int_0^{\infty} ye^{-y\theta}\,dy + \cdots + \int_0^{\infty} y^6 e^{-y\theta}\,dy\right)$$

$$= \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\left(\frac{\Gamma(1)}{\theta^1}+\frac{\Gamma(2)}{\theta^2}+\cdots+\frac{\Gamma(7)}{\theta^7}\right)$$

$$= \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\frac{1}{\theta^7}\left(\frac{\theta^7\Gamma(1)}{\theta^1}+\frac{\theta^7\Gamma(2)}{\theta^2}+\cdots+\frac{\theta^7\Gamma(7)}{\theta^7}\right)$$

$$= \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\frac{1}{\theta^7}\left(\theta^6\Gamma(1)+\theta^5\Gamma(2)+\cdots+\Gamma(7)\right)$$

$= \frac{1}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\left(\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}\right) = 1$

1) Average Mixing Opportunity Density Function 7 Gamma Distribution

Average Mixed Chance Density Function 7 Gamma distribution is a very important statistical measure and is inherent in an opportunity density function. The average or expectation can be determined as follows:

$$E(y) = \int_0^\infty y f(y) = \int_0^\infty y \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\sum_{i=0}^{7-1} y^i\, e^{-y\theta} dy$$
$$= \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\int_0^\infty y(1 + y + \cdots + y^{7-1})\, e^{-y\theta} dy$$
$$= \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\left(\int_0^\infty y\, e^{-y\theta} dy + \int_0^\infty y^2\, e^{-y\theta} dy + \cdots + \int_0^\infty y^7\, e^{-y\theta} dy\right)$$
$$= \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\left(\frac{\Gamma(2)}{\theta^2} + \frac{\Gamma(3)}{\theta^3} + \cdots + \frac{\Gamma(8)}{\theta^8}\right)$$
$$= \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\,\frac{1}{\theta^7}\left(\theta^{7-2}\Gamma(2) + \theta^{7-3}\Gamma(3) + \cdots + \theta^{7-(7+1)}\Gamma(7+1)\right) = \frac{\sum_{i=2}^{7+1}\Gamma(i)\theta^{7-i}}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}$$

2) Variation of Mixed Opportunity Density Function 7 Gamma Distribution

The 2nd moment of the Mixed Opportunity Density Function 7 Gamma distribution is a very important statistical measure, the 2nd moment is the main element in determining variation, the 2nd moment can be produced as follows:

$$E(y^2) = \int_0^\infty y^2 f(y) = \int_0^\infty y^2 \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\sum_{i=0}^{7-1} y^i\, e^{-y\theta} dy$$
$$= \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\int_0^\infty y^2 \sum_{i=0}^{7-1} y^i\, e^{-y\theta} dy$$
$$= \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\left(\int_0^\infty y^2\, e^{-y\theta} dy + \int_0^\infty y^3\, e^{-y\theta} dy + \cdots + \int_0^\infty y^{7+1}\, e^{-y\theta} dy\right)$$
$$= \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\left(\frac{\Gamma(3)}{\theta^3} + \frac{\Gamma(4)}{\theta^4} + \cdots + \frac{\Gamma(7+2)}{\theta^{7+2}}\right)$$
$$= \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\frac{1}{\theta^7}\left(\theta^{7-3}\Gamma(3) + \theta^{7-4}\Gamma(4) + \cdots + \theta^{7-(7+2)}\Gamma(7+2)\right) = \frac{\sum_{i=3}^{7+2}\Gamma(i)\theta^{7-i}}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}$$

As discussed earlier, variations can be determined by using the relationship . The Mean and Variation for the Mixed Chance Density Function 7 Gamma distribution can be determined by using the value k = 7. The average can be determined as follows:$V(y) = E(y^2) - \left(E(y)\right)^2$

$$E(y) = \frac{\sum_{i=2}^{7+1}\Gamma(i)\theta^{7-i}}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}} = \frac{\Gamma(2)\theta^{7-2}+\Gamma(3)\theta^{7-3}+\cdots+\Gamma(8)\theta^{7-8}}{\Gamma(1)\theta^{7-1}+\Gamma(2)\theta^{7-2}+\cdots+\Gamma(7)\theta^{7-7}}$$
$$= \frac{\theta^5+2\theta^4+\cdots+\frac{5040}{\theta}}{\theta^6+\theta^5+\cdots+720} = \frac{\theta^6+2\theta^5+\cdots+5040}{\theta(\theta^6+\theta^5+\cdots+720)}$$

As for variations, it is necessary to first determine the 2nd moment as follows:

$$E(y^2) = \frac{\sum_{i=3}^{7+2}\Gamma(i)\theta^{7-i}}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}} = \frac{\Gamma(3)\theta^{7-3}+\Gamma(4)\theta^{7-4}+\cdots+\Gamma(8)\theta^{7-8}+\Gamma(9)\theta^{7-9}}{\Gamma(1)\theta^{7-1}+\Gamma(2)\theta^{7-2}+\cdots+\Gamma(7)\theta^{7-7}}$$
$$= \frac{2\theta^4+6\theta^3+\cdots+5040\theta^{-1}+40320\theta^{-2}}{\theta^6+\theta^5+\cdots+720}$$
$$= \frac{\frac{2\theta^6}{\theta^2}+\frac{6\theta^5}{\theta^2}+\cdots+\frac{5040\theta}{\theta^2}+\frac{40320}{\theta^2}}{\theta^6+\theta^5+\cdots+720} = \frac{2\theta^6+6\theta^5+\cdots+5040\theta+40320}{\theta^2(\theta^6+\theta^5+\cdots+720)}$$

So the variation for the probability density function of the mixture 7 Gamma distribution can be determined as follows:

$$V(y) = E(y^2) - \left(E(y)\right)^2 = \frac{2\theta^6+6\theta^5+\cdots+5040\theta+40320}{\theta^2(\theta^6+\theta^5+\cdots+720)} - \left(\frac{\theta^6+2\theta^5+\cdots+5040}{\theta(\theta^6+\theta^5+\cdots+720)}\right)^2$$

b. Mixed Opportunity Density Function 8 Gamma Distribution

A mixed opportunity density function of 8 Gamma distributions can be generated by Using the concept of a mixed opportunity density function (2.11) as follows:

$$f(y) = \sum_{i=1}^{8} w_i\, f_i(y) = w_1 f_1(y) + w_2 f_2(y) + \cdots + w_8 f_8(y)$$

$$= \frac{\theta^7}{\theta^7+\theta^6+2\theta^5+6\theta^4+24\theta^3+120\theta^2+720\theta+5040}\ \theta e^{-y\theta} + \cdots +$$
$$\frac{5040}{\theta^7+\theta^6+2\theta^5+6\theta^4+24\theta^3+120\theta^2+720\theta+5040}\ \frac{\theta^8}{5040} y^7 e^{-y\theta}$$
$$= \frac{\theta^8}{\theta^7+\theta^6+2\theta^5+6\theta^4+24\theta^3+120\theta^2+720\theta+5040}(1+y+y^2+\cdots+y^7)e^{-y\theta}$$
$$= \frac{\theta^8}{\Gamma(1)\theta^7+\Gamma(2)\theta^6+\Gamma(3)\theta^5+\Gamma(4)\theta^4+\Gamma(5)\theta^3+\Gamma(6)\theta^2+\Gamma(7)\theta+\Gamma(8)}(1+y+\cdots+y^7)e^{-y\theta}$$
$$= \frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\left(\sum_{i=0}^{8-i} y^i\right)e^{-y\theta}$$

The mixed opportunity density function of 8 Gamma Distribution above, will be proven to meet the most important property of a jug density function, namely by proving (2.1). This proof can be done easily by understanding the integral modification form of the Gamma opportunity density function (2.10), so that the proof of this property can be done as follows: $\int_{-\infty}^{\infty} f(y)\,dy = 1$ $\int_0^\infty y^{\alpha-1}e^{-y\theta}\,dy = \frac{\Gamma(\alpha)}{\theta^\alpha}$

$$\int_0^\infty \frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\sum_{i=0}^{8-1} y^i\, e^{-y\theta}\,dy = \frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\int_0^\infty (1+y+\cdots+y^{8-1})\,e^{-y\theta}dy$$
$$= \frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\left(\int_0^\infty e^{-y\theta}\,dy + \int_0^\infty ye^{-y\theta}\,dy + \cdots + \int_0^\infty y^7e^{-y\theta}\,dy\right)$$
$$= \frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\left(\frac{\Gamma(1)}{\theta^1}+\frac{\Gamma(2)}{\theta^2}+\cdots+\frac{\Gamma(8)}{\theta^8}\right)$$
$$= \frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\frac{1}{\theta^8}\left(\frac{\theta^8\Gamma(1)}{\theta^1}+\frac{\theta^8\Gamma(2)}{\theta^2}+\cdots+\frac{\theta^8\Gamma(8)}{\theta^8}\right)$$
$$= \frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\frac{1}{\theta^8}\left(\theta^7\Gamma(1)+\theta^6\Gamma(2)+\cdots+\Gamma(8)\right)$$
$$= \frac{1}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\left(\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}\right) = 1$$

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Average Mixed Chance Density Function 8 Gamma distribution is a very important statistical measure and is inherent in an opportunity density function. The average or expectation can be determined as follows:

$$E(y) = \int_0^\infty yf(y) = \int_0^\infty y\frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\sum_{i=0}^{8-1} y^i\, e^{-y\theta}dy$$
$$= \frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\int_0^\infty y(1+y+\cdots+y^{8-1})\,e^{-y\theta}dy$$
$$= \frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{7-i}}\left(\int_0^\infty y\,e^{-y\theta}dy + \int_0^\infty y^2\,e^{-y\theta}dy + \cdots + \int_0^\infty y^8\,e^{-y\theta}dy\right)$$
$$= \frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\left(\frac{\Gamma(2)}{\theta^2}+\frac{\Gamma(3)}{\theta^3}+\cdots+\frac{\Gamma(9)}{\theta^9}\right)$$
$$= \frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\ \frac{1}{\theta^8}\left(\theta^{8-2}\Gamma(2)+\theta^{8-3}\Gamma(3)+\cdots+\theta^{8-(8+1)}\Gamma(8+1)\right) = \frac{\sum_{i=2}^{8+1}\Gamma(i)\theta^{8-i}}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}$$

2) Variation of Mixing Opportunity Density Function 8 Gamma Distribution

The 2nd moment of the Mixed Opportunity Density Function 8 Gamma distribution is a very important statistical measure, the 2nd moment is the main element in determining variation, the 2nd moment can be produced as follows:

$$E(y^2) = \int_0^\infty y^2f(y) = \int_0^\infty y^2\frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\sum_{i=0}^{8-1} y^i\, e^{-y\theta}dy$$
$$= \frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\int_0^\infty y^2\sum_{i=0}^{8-1} y^i\, e^{-y\theta}dy$$
$$= \frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\left(\int_0^\infty y^2\,e^{-y\theta}dy + \int_0^\infty y^3\,e^{-y\theta}dy + \cdots + \int_0^\infty y^{8+1}\,e^{-y\theta}dy\right)$$
$$= \frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\left(\frac{\Gamma(3)}{\theta^3}+\frac{\Gamma(4)}{\theta^4}+\cdots+\frac{\Gamma(8+2)}{\theta^{8+2}}\right)$$
$$= \frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\frac{1}{\theta^8}\left(\theta^{8-3}\Gamma(3)+\theta^{8-4}\Gamma(4)+\cdots+\theta^{8-(8+2)}\Gamma(8+2)\right) = \frac{\sum_{i=3}^{8+2}\Gamma(i)\theta^{8-i}}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}$$

As discussed earlier, variations can be determined by using the relationship . Mean and Variation for Mixed Chance Density Function 8 Gamma Distribution can be determined by using the value k = 8. The average can be determined as follows:$V(y) = E(y^2) - \left(E(y)\right)^2$

$$E(y) = \frac{\sum_{i=2}^{8+1}\Gamma(i)\theta^{8-i}}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}} = \frac{\Gamma(2)\theta^{8-2}+\Gamma(3)\theta^{8-3}+\cdots+\Gamma(9)\theta^{8-9}}{\Gamma(1)\theta^{8-1}+\Gamma(2)\theta^{8-2}+\cdots+\Gamma(8)\theta^{8-8}}$$

$$= \frac{\theta^6+2\theta^5+\cdots+\frac{40320}{\theta}}{\theta^7+\theta^6+\cdots+5040} = \frac{\theta^7+2\theta^6+\cdots+40320}{\theta(\theta^7+\theta^6+\cdots+5040)}$$

As for variations, it is necessary to first determine the 2nd moment as follows:

$$E(y^2) = \frac{\sum_{i=3}^{8+2}\Gamma(i)\theta^{8-i}}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}} = \frac{\Gamma(3)\theta^{8-3}+\Gamma(4)\theta^{8-4}+\cdots+\Gamma(9)\theta^{8-9}+\Gamma(10)\theta^{8-10}}{\Gamma(1)\theta^{8-1}+\Gamma(2)\theta^{8-2}+\cdots+\Gamma(8)\theta^{8-8}}$$

$$= \frac{2\theta^7+6\theta^6+\cdots+40320\theta+362880}{\theta^2(\theta^7+\theta^6+\cdots+5040)}$$

So the variation for the probability density function of the mixture 7 Gamma distribution can be determined as follows:

$$V(y) = E(y^2) - \left(E(y)\right)^2 = \frac{2\theta^7+6\theta^6+\cdots+40320\theta+362880}{\theta^2(\theta^7+\theta^6+\cdots+5040)} - \left(\frac{\theta^7+2\theta^6+\cdots+40320}{\theta(\theta^7+\theta^6+\cdots+5040)}\right)^2$$

3.6 Estimation of Mixed Probability Density Function Parameters of 7 Gamma Distribution and 8 Gamma Distribution Using Maximum Likelihood

Parameter estimation for the mixed chance density function of 7 Gamma distribution will be generated by a numerical iteration process, where the Newton rhapson method will be used in this case. The probability density function of the mixture 7 gamma distribution is as follows:

$$f(y) = \frac{\theta^7}{\Gamma(1)\theta^6+\Gamma(2)\theta^5+\Gamma(3)\theta^4+\Gamma(4)\theta^3+\Gamma(5)\theta^2+\Gamma(6)\theta+\Gamma(7)}(1+y+\cdots+y^6)e^{-y\theta}, y>0, \theta>0$$

To simplify the writing of the density function above, it will be partitioned into several parts, as follows:

$$S(y) = 1+y+\cdots+y^6$$
$$D(\theta) = \Gamma(1)\theta^6+\Gamma(2)\theta^5+\Gamma(3)\theta^4+\Gamma(4)\theta^3+\Gamma(5)\theta^2+\Gamma(6)\theta+\Gamma(7)$$

So that the probability density function of the mixture of 7 gamma distributions is simplified into:

$$f(y) = \frac{\theta^7}{D(\theta)}S(y)e^{-y\theta}, y>0, \theta>0$$

The Likelihood function can be derived based on the probability density function above with equation (2.12) as follows:

$$L(\theta) = \prod_{i=1}^{n} f(y_i) = \left(\frac{\theta^7}{D(\theta)}\right)^n \prod_{i=1}^{n} S(y_i)\ exp\left(-\theta\sum_{i=1}^{n} y_i\right)$$

The likelihood log function can be derived with equation (2.13) as follows:

$$l(\theta) = \log\left(L(\theta)\right) = n\left(7\log(\theta) - \log\left(D(\theta)\right)\right) + \log\left(\sum_{i=1}^{n} S(y_i)\right) - \theta\sum_{i=1}^{n} y_i$$

The parameter estimation process will be carried out using the numerical iteration method, namely the Newton Rhapson method, where this method requires the first and second derivatives for the likelihood log function which is successively notated as , where it can be derived as follows. To facilitate this, it is necessary to derive first both the first derivative and the second derivative of , this is because the form is directly related to , so that the first derivative and the second derivative can be given as follows:$l'(\theta)\ dan\ l''(\theta)\ D(\theta)\theta D'(\theta) D''(\theta)$

$$D'(\theta) = 6\Gamma(1)\theta^5+5\Gamma(2)\theta^4+4\Gamma(3)\theta^3+3\Gamma(4)\theta^2+2\Gamma(5)\theta+\Gamma(6)$$
$$D''(\theta) = 30\Gamma(1)\theta^4+20\Gamma(2)\theta^3+12\Gamma(3)\theta^2+6\Gamma(4)\theta+2\Gamma(5)$$

Using the derivatives of the above form, the first derivative and the second derivative of the probability log function can be generated as follows:

$$l'(\theta) = n\frac{7}{\theta} - n\frac{D'(\theta)}{D(\theta)} - \sum_{i=1}^{n} y_i$$

$$l''(\theta) = -\frac{7n}{\theta^2} - n\frac{D''(\theta)D(\theta) - \left(D'(\theta)\right)^2}{D(\theta)^2}$$

Parameter estimation with a numerical iteration process uses the Newton Rhapson method, For the initial value of the given parameters, the parameter iteration can be generated manually or using tools such as computers. $\theta\left(\hat{\theta}_{k=0}\right)\hat{\theta}_{k+1}$

The dataset shown in table 4 will be used to estimate the parameters of the mixed chance density function of 7 gamma distributions up to 3 iterations.

Table 4. Set Data

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958
1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179
2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382
2.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554
2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726
2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.858	

The initial value to be used in this study is 0.40728388 so some important values needed to get parameter estimates in the 1st iteration are as follows:$\left(\hat{\theta}_{k=0}\right)$

$$n = 69, \qquad \sum_{i=1}^{n} y_i = 169{,}415, \qquad D(\theta) = 773{,}33135631,$$
$$D'(\theta) = 143{,}28077484, \qquad D''(\theta) = 68{,}82003724,$$
$$l'(\theta) = 1003{,}70586369, \qquad l''(\theta) = -2915{,}51232903$$

So the estimated parameter of iteration 1 is:

$$\hat{\theta}_1 = \hat{\theta}_0 - \frac{l'\left(\hat{\theta}_k\right)}{l''\left(\hat{\theta}_k\right)}$$
$$\hat{\theta}_1 = 0{,}40728388 - \frac{1003{,}70586369}{-2915{,}51232903} = 0{,}75154786$$

For the 2nd iteration, it can be done in the same way by first using the initial iteration, which is the value of the 1st iteration obtained earlier. The important values to get the 2nd iteration are as follows:$(0{,}75154786)$

$$n = 69, \qquad \sum_{i=1}^{n} y_i = 169{,}415, \qquad D(\theta) = 827{,}325462,$$
$$D'(\theta) = 171{,}268162, \qquad D''(\theta) = 147{,}213453,$$
$$l'(\theta) = 418{,}736270, \qquad l''(\theta) = -785{,}362071$$

So the estimated parameters of iteration 2 are:

$$\hat{\theta}_2 = \hat{\theta}_1 - \frac{l'\left(\hat{\theta}_k\right)}{l''\left(\hat{\theta}_k\right)}$$
$$\hat{\theta}_2 = 0{,}75154786 - \frac{418{,}736270}{-785{,}362071} = 1{,}28446902$$

For the 3rd iteration, it can be done in the same way by first using the initial iteration, which is the value of the 2nd iteration obtained earlier. The important values to get the 3rd iteration are as follows:$(1{,}28446902)$

$$n = 69, \qquad \sum_{i=1}^{n} y_i = 169{,}415, \qquad D(\theta) = 939{,}8795,$$
$$D'(\theta) = 262{,}89402445, \qquad D''(\theta) = 257{,}88264739,$$
$$l'(\theta) = 187{,}31585300, \qquad l''(\theta) = -306{,}28568869$$

So the estimated parameters of iteration 3 are:

$$\hat{\theta}_3 = \hat{\theta}_2 - \frac{l'(\hat{\theta}_k)}{l''(\hat{\theta}_k)}$$

$$\hat{\theta}_3 = 1{,}28446902 - \frac{187{,}31585300}{-306{,}28568869} = 1{,}89604135$$

3.7 Parameter estimation for the 8 Gamma distribution mixed chance density function

Can be produced by first providing the following form of opportunity density function:

$$f(y) = \frac{\theta^8}{\Gamma(1)\theta^7 + \Gamma(2)\theta^6 + \Gamma(3)\theta^5 + \Gamma(4)\theta^4 + \Gamma(5)\theta^3 + \Gamma(6)\theta^2 + \Gamma(7)\theta + \Gamma(8)}(1 + y + \cdots + y^7)e^{-y\theta}, y > 0, \theta > 0$$

To simplify the writing of the density function above, it will be partitioned into several parts, as follows:

$$S(y) = 1 + y + \cdots + y^7$$
$$D(\theta) = \Gamma(1)\theta^7 + \Gamma(2)\theta^6 + \Gamma(3)\theta^5 + \Gamma(4)\theta^4 + \Gamma(5)\theta^3 + \Gamma(6)\theta^2 + \Gamma(7)\theta + \Gamma(8)$$

So that the probability density function of the mixture 8 gamma distribution is simplified to:

$$f(y) = \frac{\theta^8}{D(\theta)}S(y)e^{-y\theta}, y > 0, \theta > 0$$

The Likelihood function can be derived based on the probability density function above with equation (2.12) as follows:

$$L(\theta) = \prod_{i=1}^{n} f(y_i) = \left(\frac{\theta^8}{D(\theta)}\right)^n \prod_{i=1}^{n} S(y_i)\ exp\left(-\theta\sum_{i=1}^{n} y_i\right)$$

The likelihood log function can be derived with equation (2.13) as follows:

$$l(\theta) = \log(L(\theta)) = n(8\log(\theta) - \log(D(\theta))) + \log\left(\sum_{i=1}^{n} S(y_i)\right) - \theta\sum_{i=1}^{n} y_i$$

The parameter estimation process will be carried out using the numerical iteration method, namely the Newton-Raphson method, where this method requires the first and second derivatives for the likelihood log function, which is successively notated as, where it can be derived as follows. To facilitate this, it is necessary to derive first both the first derivative and the second derivative of, this is because the form is directly related to, so that the first derivative and the second derivative can be given as follows:$l'(\theta)\ dan\ l''(\theta)\ D(\theta)\theta D'(\theta)D''(\theta)$

$$D'(\theta) = 7\Gamma(1)\theta^6 + 6\Gamma(2)\theta^5 + 5\Gamma(3)\theta^4 + 4\Gamma(4)\theta^3 + 3\Gamma(5)\theta^2 + 2\Gamma(6)\theta + \Gamma(7)$$
$$D''(\theta) = 42\Gamma(1)\theta^5 + 30\Gamma(2)\theta^4 + 20\Gamma(3)\theta^3 + 12\Gamma(4)\theta^2 + 6\Gamma(5)\theta + 2\Gamma(6)\theta$$

Using the derivatives of the above form, the first derivative and the second derivative of the probability log function can be generated as follows:

$$l'(\theta) = n\frac{8}{\theta} - n\frac{D'(\theta)}{D(\theta)} - \sum_{i=1}^{n} y_i$$

$$l''(\theta) = -\frac{8n}{\theta^2} - n\frac{D''(\theta)D(\theta) - (D'(\theta))^2}{D(\theta)^2}$$

Parameter estimation with a numerical iteration process uses the Newton Rhapson method (2.14), For the initial value of the given parameters, the parameter iteration can be generated manually or using tools such as computers. $\theta(\hat{\theta}_{k=0})\hat{\theta}_{k+1}$

The same dataset as in the previous discussion will be used to estimate the parameters of the mixed chance density function of 8 gamma distributions up to 3 iterations.

The initial value to be used in this study is 0.40728388 so some important values needed to get parameter estimates in the 1st iteration are as follows:$(\hat{\theta}_{k=0})$

$$n = 69, \quad \sum_{i=1}^{n} y_i = 169{,}415, \quad D(\theta) = 5354{,}95, \quad D'(\theta) = 831{,}685$$
$$D''(\theta) = 314{,}589, \quad l'(\theta) = 1175{,}188, \quad l''(\theta) = -3329{,}54$$

So the estimated parameter of iteration 1 is:

$$\hat{\theta}_1 = \hat{\theta}_0 - \frac{l'(\hat{\theta}_k)}{l''(\hat{\theta}_k)}$$

$$\hat{\theta}_1 = 0{,}40728388 - \frac{1175{,}188}{-3329{,}54} = 0{,}7602839$$

For the 2nd iteration, it can be done in the same way by first using the initial iteration, which is the value of the 1st iteration obtained earlier. The important values to get the 2nd iteration are as follows:(0,7602839)

$$n = 69,\ \ \sum_{i=1}^{n} y_i = 169{,}415,\ \ D(\theta) = 5670{,}1681,\ \ D'(\theta) = 960{,}8509,$$

$$D''(\theta) = 429{,}3706\,, \quad l'(\theta) = 544{,}9341, \quad l''(\theta) = -958{,}35$$

So the estimated parameters of iteration 2 are:

$$\hat{\theta}_2 = \hat{\theta}_1 - \frac{l'(\hat{\theta}_k)}{l''(\hat{\theta}_k)}$$

$$\hat{\theta}_2 = 0{,}7602839 - \frac{544{,}9341}{-958{,}35} = 1{,}3289$$

For the 3rd iteration, it can be done in the same way by first using the initial iteration, which is the value of the 2nd iteration obtained earlier. The important values to get the 3rd iteration are as follows:(1,3289)

$$n = 69, \quad \sum_{i=1}^{n} y_i = 169{,}415, \quad D(\theta) = 6306{,}0001,$$

$$D'(\theta) = 1316{,}9, \quad D''(\theta) = 919{,}7, \quad l'(\theta) = 231{,}7, \quad l''(\theta) = -319{,}7$$

So the estimated parameters of iteration 3 are:

$$\hat{\theta}_3 = \hat{\theta}_2 - \frac{l'(\hat{\theta}_k)}{l''(\hat{\theta}_k)}$$

$$\hat{\theta}_3 = 1{,}3289 - \frac{231{,}7}{-319{,}7} = 2.0537$$

4. Conclusion

Based on the results of the research and discussion in Chapter IV, the following conclusions were obtained: Obtain a new opportunity model using a mixture of 7 with 1 parameter of the gamma opportunity density function: $f(y) = \frac{\theta^7}{\sum_{i=1}^{7}\Gamma(i)\theta^{7-i}}\left(\sum_{i=0}^{7-i} y^i\right)e^{-y\theta}$and a new opportunity model using a mixture of 8 with 1 parameter of the gamma opportunity density function : $f(y) = \frac{\theta^8}{\sum_{i=1}^{8}\Gamma(i)\theta^{8-i}}\left(\sum_{i=0}^{8-i} y^i\right)e^{-y\theta}$. Obtaining other important characteristics from the new opportunity model model using a mixture of 7 and 8 with 1 parameter of the gamma chance density function which is a value of 1. Obtaining proof of new opportunity models with existing 1-parameter opportunity models : $f(y) = \frac{\theta^k}{\sum_{i=1}^{k}\Gamma(i)\theta^{k-1}}(1 + \sum_{i=1}^{k-1} x^i)e^{-\theta x}$. The authors suggest that further research use the pattern regularity of the mixed chance density functions 7 and 8 gamma distributions to generate the next new chance density functions.

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