Jurnal Sains Matematika dan Statistika
p-ISSN: 2460-4542
e-ISSN: 2615-8663

Spectrum of the m-Copy Cycle Graphs mC_3, mC_4, mC_5 and mC_6

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Submitted : 16 April 2025 Accepted : 20 Januari 2026 Published : 30 Januari 2026

Abstract

The adjacency matrix $A(G)$ of a simple graph with n vertices is a matrix of size $n \times n$ with the ijth entry (ith row and jth column) having the value 0 or 1. The adjacency spectrum of a graph is denoted $Spec\ (G)$ is a matrix of size $2 \times p$, with p representing the number of different eigenvalues of $A(G)$. In this research, the author is looking for the formulation of the adjacency spectrum pattern of the circle graphs mC_3, mC_4, mC_5 dan mC_6. Formulating the pattern begins by determining $A(G)$ of each graph for the values $n = 3, 4, 5$ dan 6, then looking for the eigenvalues of $A(G)$ and their multiplicity. From these results, the spectrum of each graph is formulated into a theorem and its correctness is proven.

Keywords: *Adjacency matrix, Circle graph, Block matrix, Adjacent Graph spectrum.*

1. Introduction

The study of relationships among objects that can be modeled as vertices and edges, enabling the analysis of network structures and properties, is known as graph theory [1]. A graph is defined as a mathematical structure consisting of a set of vertices and a set of edges, where the edges connect certain pairs of vertices [2]. In graph theory, one of the main aspects of interest is the graph spectrum, namely the set of eigenvalues of the graph's adjacency matrix [3]. The graph spectrum contains important information about the graph's structure and algebraic properties, which can be used for various analyses and practical applications.

One interesting topic in graph theory is the study of cycle graphs and their copies. A cycle graph is a directed graph that represents cyclic relations, where each vertex is incident to exactly one incoming edge and one outgoing edge, forming a closed circuit [4]. In this study, the author focuses on the cycle graphs C_3, C_4, C_5,and C_6, which have 3, 4, 5, and 6 vertices, respectively. Research on copies of these graphs is important because it can produce larger and more complex graph structures, which often arise in various practical applications.

The spectrum of the m-copy graphs mC_3, mC_4, mC_5, and mC_6 is a research area that offers many important insights. By analyzing the spectra of these graphs, various algebraic properties of the graphs can be understood. The graph spectrum can provide information about graph stability, the existence of certain subgraphs, and other important parameters that can be used in network design and analysis. In addition, the graph spectrum also has applications in fields such as control theory, signal analysis, and optimization [5].

Valette in Switzerland (2017) studied the spectrum of the graph RX, which is obtained from a connected graph X by associating a new vertex to each edge of X and connecting the endpoints of each edge of X to the corresponding new vertex [6]. The result obtained is:

$$P_{RX}(\lambda) = 2^{-n}(1-\lambda)^{m-n}\left(\frac{3}{2}-\lambda\right)^{n} P_X(2\lambda)$$

The spectrum of the complete graph K_n was studied by Selvia, Narwen, and Zulakmal [7]. The results of their research are presented in Table 1 below:

Table 1. Spectrum of the Complete Graph K_n

Matrix	Eigen Value	Multiplicity
Adjacency	$(n-1)\quad 1$	$1\quad (n-1)$
Laplacian	$n\quad 0$	$(n-1)\quad 1$
Signless Laplacian	$2(n-1)\quad (n-2)$	$1\quad (n-1)$
Normalized Laplacian	$\frac{n}{(n-1)}\quad 0$	$(n-1)\quad 1$
Seidel Adjacency	$-(n-1)\quad 1$	$1\quad (n-1)$

Another study was conducted by Triyani, who succeeded in finding the general form of the spectrum of the prism graph $P_{(2,S)}$ [8]. The result obtained is as follows:

$$Spec\ P_{(2,S)} = \begin{pmatrix} \lambda_0 & \lambda_{1,2s-1} & \lambda_{1,2s-2} & \cdots & \lambda_{1,2s+1} & \lambda_s \\ 1 & 2 & 2 & \cdots & 2 & 1 \end{pmatrix}$$

A recent study in 2022 was conducted by Agustina, Kusumastuti, and Fran [9]. In that study, they identified patterns of the adjacency spectra of the star graph, crown graph, and ladder graph as follows:

a. The adjacency spectrum of the star graph S_n is $Spec\ (S_n) = \begin{bmatrix} -\sqrt{n} & 0 & \sqrt{n} \\ 1 & n-1 & 1 \end{bmatrix}, n \geq 2.$

b. The adjacency spectrum of the crown graph S_n^0 is $Spec\ (S_n^0) = \begin{bmatrix} n-1 & 1 & -1 & 1-n \\ 1 & n-1 & n-1 & 1 \end{bmatrix}, n \geq 2.$

c. The adjacency spectrum of the ladder graph L_n is $Spec\ (L_n) = \begin{bmatrix} 1+2\cos\left(\frac{n\pi}{n+1}\right) & \cdots & -1+2\cos\left(\frac{n\pi}{n+1}\right) \\ 1 & \cdots & 1 \end{bmatrix}, n \geq 2.$

Based on these previous studies, the author is interested in examining the general form of the m-copy cycle graphs mC_3, mC_4, mC_5, and mC_6. This research aims to explore in depth the spectra of the m-copy cycle graphs mC_3, mC_4, mC_5, and mC_6. It will discuss how the eigenvalues of the adjacency matrix change when the cycle graphs C_3, C_4, C_5, and C_6 are replicated into m copies. In addition, the patterns that emerge in the spectra of these copy graphs will be investigated. By understanding the spectra of these copy graphs, this study is expected to make a significant contribution to the development of graph theory and its applications in various fields.

2. Research Methodology

The method used in this study is a literature review. This research is conducted to examine the general form of the spectra of the cycle graphs mC_3, mC_4, mC_5, and mC_6. The steps in this study are as follows:

1. Drawing the cycle graphs mC_3, mC_4, mC_5, and mC_6.
2. Determining the adjacency matrices of the cycle graphs mC_3, mC_4, mC_5, and mC_6.
3. Determining the characteristic polynomials of the cycle graphs mC_3, mC_4, mC_5, and mC_6 in order to obtain their eigenvalues and algebraic multiplicities.
4. Formulating the general form of the spectra of the cycle graphs mC_3, mC_4, mC_5, and mC_6.

2.1 Graph

Definition 2.1 [10] A graph G is defined as an ordered pair (V, E), where V is a non-empty set of vertices (or nodes) and E is a set of edges that connect pairs of vertices.

2.2 Cycle Graph

A cycle graph is a connected graph in which every vertex has degree 2. A cycle graph with n vertices is denoted by C_n. In other words, a cycle graph is a graph that contains a cycle, namely a path that starts and ends at the same vertex through a sequence of edges, without visiting the same vertex more than once, except for the starting and ending vertex [11].

In the context of cycle graphs, each vertex is connected to exactly two other edges. Therefore, a cycle graph is often represented as a simple circuit or polygon, where the vertices correspond to the corners and the edges represent the graph edges connecting those vertices [4].

2.3. Graph Copies

A graph copy can be obtained by replicating all vertices and edges of the original graph [12]. In other words, a copy of a graph is a graph that is isomorphic to the original graph, as explained in Definition 2.2 below:

Definition 2.2 [10] Two graphs $G_1$and G_2 are said to be isomorphic ($G_1 \cong G_2$) if there exists a one-to-one correspondence between the vertices and edges of both graphs such that, if an edge e is incident to vertices $u$and v in G_1, then the corresponding edge e' must also be incident to the corresponding vertices u' and v' in G_2.

2.4.Matrix

A matrix is a rectangular arrangement of numbers. The numbers in the arrangement are called the entries of the matrix [13]. The general form of a matrix is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where A is the name of the matrix, m is the number of rows, n is the number of columns, and $m \times n$ is the order (dimension) of the matrix.

2.5. Block Matrix

A block matrix can be visualized as an original matrix with horizontal and vertical lines that break it (partition it) into a collection of smaller matrices [13].

Definition 2.3 [14] A block matrix, or partitioned matrix, is a matrix that is partitioned into several smaller matrices by inserting horizontal and vertical lines between its rows and columns. The smaller matrices resulting from the partition are called submatrices.

3. Block Diagonal Matrix

A block diagonal matrix is a block matrix that is square, in which the main diagonal blocks are square matrices and all off-diagonal blocks are zero matrices [15]. The general form of a block diagonal matrix is:

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$$

where A_k is a square matrix for each $k = 1,2,\dots,n$.

Theorem 2.1 [16] If $A_1, A_2, \dots, A_k$are square matrices on the diagonal, then the determinant of matrix Acan be written as:

$$\det(\mathrm{A}) = \prod_{i=1}^{k} \det(A_i)$$

This can also be expressed as:

$$\det(\mathrm{A}) = \det(A) = \det(A_1) \times \det(A_2) \times \dots \times \det(A_k)$$

4. Determinant

The determinant function is denoted by det. Let A be a square matrix; then $\det(A)$ can be formulated as the sum of all elementary products of A [17].

Theorem 2.2 [17] Suppose A is a square matrix. The minor of a_{ij}, denoted by M_{ij}, is the determinant of the submatrix that remains after deleting the i-th row and the j-th column. The cofactor of a_{ij} is denoted by C_{ij} and is given by $C_{ij} = (-1)^{i+j} M_{ij}$. For $1 \le i \le n$ and $1 \le j \le n$, the determinant can be written as:

$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$ (Cofactor expansion along the i-th row).

$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{mj}C_{mj}$ (Cofactor expansion along the j-th column).

5. Characteristic Polynomial

The determinant of a matrix yields a characteristic polynomial, and the eigenvalues are obtained from the roots of this characteristic polynomial. This is described in the following definition:

Definition 2.3 [18] Let A be an $n \times n$ matrix. A scalar λ is an eigenvalue of A if there exists a nonzero vector $x_{n\times 1} \neq 0$ such that $Ax = \lambda x$. A vector satisfying this equation is called an eigenvector corresponding to the eigenvalue λ. The characteristic polynomial of A is the

DOI : https://dx.doi.org/10.24014/jsms.v12i1.36489

polynomial $\det(\lambda I - A)$. The roots of the characteristic polynomial are the eigenvalues of the matrix A.

6. Spektrum Graf

To determine the spectrum of a graph, it is necessary to introduce terminology related to the adjacency matrix, eigenvalues, and multiplicities. Further explanation is provided in Definition 2.4.

Definisi 2.4 [19] If G is a graph with n vertices $v_1, v_2, \dots, v_n$, then the adjacency matrix of G is an $n \times n$ matrix $A(G) = (a_{ij})$, where a_{ij} is the number of edges connecting vertices v_i and v_j.

The spectrum of a graph G is the list of eigenvalues of the adjacency matrix of G together with their multiplicities [3]. The graph spectrum can be obtained through matrix operations, such as computing determinants to derive the characteristic polynomial and its eigenvalues. Spectral graph analysis has many applications in science and technology, including studying geometric properties of a network related to the Cheeger constant, analyzing signals on graphs, and examining features of random walks on graphs via stochastic transition matrices [20].

Theorem 2.3 [20] If the distinct eigenvalues of the adjacency matrix are $\lambda_0 > \lambda_1 > \cdots > \lambda_{n-1}$ with multiplicities $m(\lambda_0), m(\lambda_1), \dots, m(\lambda_{n-1})$, then it can be written as:

$$Spec\ G = \begin{bmatrix} \lambda_0 & \lambda_1 & \dots & \lambda_{n-1} \\ m(\lambda_0) & m(\lambda_1) & \dots & m(\lambda_{n-1}) \end{bmatrix}.$$

3. Result dan Discussion

3.1. Spectrum of the m-Copy Cycle Graph mC_3

The first step in determining the general form of the spectrum of the m-copy cycle graph mC_3 is to draw the graph. The following figure shows the general form of the m-copy cycle graph mC_3.

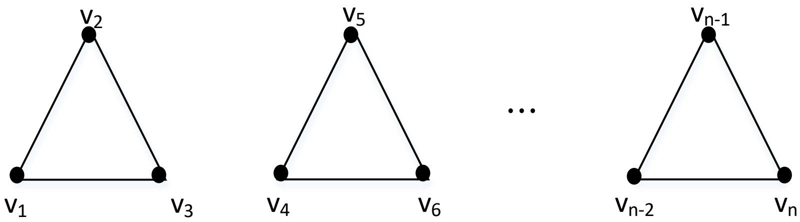

Figure 1. General form of the m-copy cycle graph mC_3

Next, based on Theorem 2.2, the general form of the characteristic polynomial of the m-copy cycle graph mC_3 is obtained, which serves as the initial step in determining the spectrum of this copy graph. Table 2 presents the characteristic polynomials of the m-copy cycle graph mC_3 for $n = 1,2,3,4,5, \dots, m$.

Table 2. Characteristic polynomial of the m-copy cycle graph mC_3

Cycle Graph mC_3	Characteristic Polynomial
$1C_3$	$(\lambda - 2)(\lambda + 1)^2$

DOI : https://dx.doi.org/10.24014/jsms.v12i1.36489

$2C_3$	$((\lambda-2)(\lambda+1)^2)^2$
$3C_3$	$((\lambda-2)(\lambda+1)^2)^3$
$4C_3$	$((\lambda-2)(\lambda+1)^2)^4$
$5C_3$	$((\lambda-2)(\lambda+1)^2)^5$
$\vdots$	$\vdots$
mC_3	$((\lambda-2)(\lambda+1)^2)^m$

From Table 2, Theorem 3.1 is obtained as follows:

Theorem 3.1

The spectrum of the m-copy cycle graph mC_3 is:

$$Spec\ mC_3 = \begin{pmatrix} 2 & -1 \\ m & 2m \end{pmatrix}$$

Proof.

The adjacency matrix of the m-copy cycle graph mC_3 is:

$$A = \begin{bmatrix} 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \end{bmatrix}$$

Based on matrix A, the eigenvalues and eigenvectors of this block diagonal matrix can be determined by solving $\det(A) = 0$. Thus, we obtain the following matrix:

$$\mathrm{A} - \lambda I = \begin{bmatrix} \lambda & -1 & -1 & \cdots & 0 & 0 & 0 \\ -1 & \lambda & -1 & \cdots & 0 & 0 & 0 \\ -1 & -1 & \lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & \lambda & -1 \\ 0 & 0 & 0 & \cdots & -1 & -1 & \lambda \end{bmatrix}$$

Let $B = \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix}$, then $\mathrm{A} - \lambda I = \begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B \end{pmatrix}$. By Theorem 2.1, we have:

$$det(A-\lambda I) = det(B) \times det(B) \times \ldots \times det(B)$$

Next, using the cofactor expansion method as in Theorem 2.2, the determinant of matrix $B$is computed by expanding along the first row, yielding:

$$det\ (B) = (\lambda-2)(\lambda+1)^2$$

Since

$$det(A-\lambda I) = (\det(B))^m$$

it follows that:

$$det(A-\lambda I) = ((\lambda-2)(\lambda+1)^2)^m$$
$$det(A-\lambda I) = (\lambda-2)^m((\lambda+1)^2)^m$$
$$det(A-\lambda I) = (\lambda-2)^m(\lambda+1)^{2m}$$

Based on Definition 2.3, the eigenvalues are the roots of the characteristic polynomial. Therefore, the eigenvalues are:

$$\lambda_1 = 2 \text{ dan } \lambda_2 = -1$$

From these eigenvalues, and using Theorem 2.3, the multiplicities are $m(2) = m$and $m(-1) = 2m$. Hence, the spectrum can be written as:

$$Spec\ mC_3 = \begin{pmatrix} 2 & -1 \\ m & 2m \end{pmatrix}.$$

3.3. Spectrum of the m-Copy Cycle Graph mC_4

The first step in determining the general form of the spectrum of the m-copy cycle graph mC_4 is to draw the graph. The following figure shows the general form of the m-copy cycle graph mC_4.

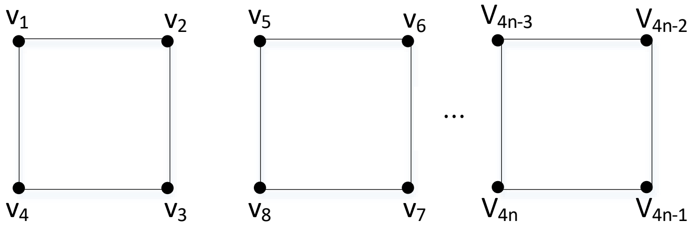

Figure 2. General form of the m-copy cycle graph mC_4

Next, based on Theorem 2.2, the general form of the characteristic polynomial of the m-copy cycle graph mC_4 is obtained, which serves as the initial step in determining the spectrum of this copy graph. Table 3 presents the characteristic polynomials of the m-copy cycle graph mC_4 for $n = 1,2,3,4,5, \dots, m$.

Table 3. Characteristic polynomial of the m-copy cycle graph mC_4

Cycle Graph mC_4	**Characteristic Polynomial**
$1C_4$	$\lambda^4 - 4\lambda^2$
$2C_4$	$(\lambda^4 - 4\lambda^2)^2$
$3C_4$	$(\lambda^4 - 4\lambda^2)^3$
$4C_4$	$(\lambda^4 - 4\lambda^2)^4$
$5C_4$	$(\lambda^4 - 4\lambda^2)^5$
⋮	⋮
mC_4	$(\lambda^2(\lambda - 2)(\lambda + 2))^m$

From Table 3, Theorem 3.2 is obtained as follows:

Theorem 3.2

The spectrum of the m-copy cycle graph mC_4 is:

$$Spec\ mC_4 = \begin{pmatrix} -2 & 0 & 2 \\ m & 2m & m \end{pmatrix}$$

Proof.

The adjacency matrix of the m-copy cycle graph mC_4 is:

DOI : https://dx.doi.org/10.24014/jsms.v12i1.36489

$$C = \begin{bmatrix} 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 \end{bmatrix}$$

Based on matrix C, the eigenvalues and eigenvectors of this block diagonal matrix can be determined by solving $\det(C) = 0$. Thus, we obtain the following matrix:

$$\mathrm{C} - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -\lambda & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -\lambda & 1 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -\lambda & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -\lambda & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & -\lambda \end{bmatrix}$$

Let $D = \begin{bmatrix} -\lambda & 1 & 0 & 1 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 1 \\ 1 & 0 & 1 & -\lambda \end{bmatrix}$, then $\mathrm{C} - \lambda I = \begin{pmatrix} D & 0 & \cdots & 0 \\ 0 & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D \end{pmatrix}$.

Based on Theorem 2.1, we have:

$$det(C - \lambda I) = det(D) \times det(D) \times \ldots \times det(D)$$

Next, using the cofactor expansion method as in Theorem 2.2, the determinant of matrix D is computed by expanding along the first row, yielding:

$$det\ (D) = \lambda^4 - 4\lambda^2$$

Since

$$det(C - \lambda I) = (\det(D))^m$$

it follows that:

$$det(C - \lambda I) = (\lambda^4 - 4\lambda^2)^m$$
$$det(C - \lambda I) = (\lambda^2(\lambda - 2)(\lambda + 2))^m$$
$$det(C - \lambda I) = (\lambda^{2m}(\lambda - 2)^m(\lambda + 2)^m).$$

Based on Definition 2.3, the eigenvalues are the roots of the characteristic polynomial. Therefore, the eigenvalues are:

$$\lambda_1 = 0, \lambda_2 = -2 \text{ and } \lambda_3 = 2.$$

From these eigenvalues, and using Theorem 2.3, the multiplicities are $m(-2) = m$, $m(0) = 2m$, and $m(2) = m$. Hence, the spectrum can be written as:

$$Spec\ mC_4 = \begin{pmatrix} -2 & 0 & 2 \\ m & 2m & m \end{pmatrix}.$$

3.4. Spectrum of the m-Copy Cycle Graph mC_5

The first step in determining the general form of the spectrum of the m-copy cycle graph mC_5 is to draw the graph. The following figure shows the general form of the m-copy cycle graph mC_5.

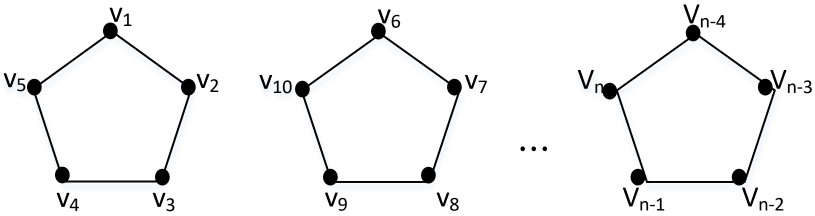

Figure 3. General form of the m-copy cycle graph mC_5

Next, based on Theorem 2.2, the general form of the characteristic polynomial of the m-copy cycle graph mC_5 is obtained, which serves as the initial step in determining the spectrum of this copy graph. Table 4 presents the characteristic polynomials of the m-copy cycle graph mC_5 for $n = 1,2,3,4,5,\dots,m$.

Table 4. Characteristic polynomial of the m-copy cycle graph mC_5

Cycle Graph mC_4	Characteristic Polynomial
$1C_5$	$(\lambda-2)(\lambda^2+\lambda-1)^2$
$2C_5$	$((\lambda-2)(\lambda+1)^2)^2$
$3C_5$	$((\lambda-2)(\lambda+1)^2)^3$
$4C_5$	$((\lambda-2)(\lambda+1)^2)^4$
$5C_5$	$((\lambda-2)(\lambda+1)^2)^5$
$\vdots$	$\vdots$
mC_5	$((\lambda-2)(\lambda+1)^2)^6$

From Table 4, Theorem 3.3 is obtained as follows:

Theorem 3.3

The spectrum of the m-copy cycle graph mC_5 is:

$$spec\ mC_5 = \begin{pmatrix} 2 & -\frac{1}{2}+\frac{1}{2}\sqrt{5} & -\frac{1}{2}-\frac{1}{2}\sqrt{5} \\ m & 2m & 2m \end{pmatrix}$$

Proof.

The adjacency matrix of the m-copy cycle graph mC_5 is:

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

DOI : https://dx.doi.org/10.24014/jsms.v12i1.36489

Based on matrix E, the eigenvalues and eigenvectors of this block diagonal matrix can be determined by solving $\det(E) = 0$. Thus, we obtain the following matrix:

$$\mathrm{E} - \lambda I = \begin{bmatrix} \lambda & -1 & 0 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ -1 & \lambda & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & \lambda & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & \lambda & -1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & \lambda & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & \lambda & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & \lambda & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & -1 & \lambda \end{bmatrix}$$

Let $F = \begin{bmatrix} \lambda & -1 & 0 & 0 & -1 \\ -1 & \lambda & -1 & 0 & 0 \\ -1 & -1 & \lambda & -1 & 0 \\ 0 & 0 & -1 & \lambda & -1 \\ -1 & 0 & 0 & -1 & \lambda \end{bmatrix}$, then $\mathrm{E} - \lambda I = \begin{pmatrix} F & 0 & \cdots & 0 \\ 0 & F & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F \end{pmatrix}$. By Theorem 2.1:

$$det(E - \lambda I) = det(F) \times det(F) \times \ldots \times det(F)$$

Next, using the cofactor expansion method as in Theorem 2.2, the determinant of matrix $F$is computed by expanding along the first row, yielding:

$$\det\ (F) = (\lambda - 2)(\lambda^2 + \lambda - 1)^2$$

Since

$$det(E - \lambda I) = (\det(F))^m$$

it follows that:

$$det(E - \lambda I) = (\lambda - 2)(\lambda^2 + \lambda - 1)^2$$
$$det(E - \lambda I) = \left((\lambda - 2)(\lambda^2 + \lambda - 1)^2\right)^m$$
$$det(E - \lambda I) = (\lambda - 2)^m(\lambda^2 + \lambda - 1)^{2m}.$$

Based on Definition 2.3, the eigenvalues are the roots of the characteristic polynomial. Therefore, the eigenvalues are:

$$\lambda_1 = 2\,, \lambda_2 = -\frac{1}{2} + \frac{1}{2}\sqrt{5} \text{ dan } \lambda_3 = -\frac{1}{2} - \frac{1}{2}\sqrt{5}$$

From these eigenvalues, and using Theorem 2.3, the multiplicities are $m(2), 2m\left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right), 2m\left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)$. Hence, the spectrum can be written as:

$$Spec\ mC_5 = \begin{pmatrix} 2 & -\frac{1}{2} + \frac{1}{2}\sqrt{5} & -\frac{1}{2} - \frac{1}{2}\sqrt{5} \\ m & 2m & 2m \end{pmatrix}.$$

3.5. Spectrum of the m-Copy Cycle Graph mC_6

The first step in determining the general form of the spectrum of the m-copy cycle graph mC_6 is to draw the graph. The following figure shows the general form of the m-copy cycle graph mC_6.

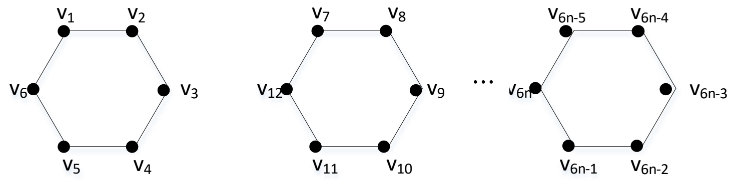

Figure 4. General form of the m-copy cycle graph mC_6

Next, based on Theorem 2.2, the general form of the characteristic polynomial of the m-copy cycle graph mC_6 is obtained, which serves as the initial step in determining the spectrum of this copy graph. Table 5 presents the characteristic polynomials of the m-copy cycle graph mC_6 for $n = 1,2,3,4,5, \dots, m$.

Table 5. Characteristic polynomial of the m-copy cycle graph mC_6

Copy Cycle Graph mC_6	Characteristic Polynomial
$1C_6$	$\lambda^6 - 6\lambda^4 + 9\lambda^2 - 4$
$2C_6$	$(\lambda^6 - 6\lambda^4 + 9\lambda^2 - 4)^2$
$3C_6$	$(\lambda^6 - 6\lambda^4 + 9\lambda^2 - 4)^3$
$4C_6$	$(\lambda^6 - 6\lambda^4 + 9\lambda^2 - 4)^4$
$5C_6$	$(\lambda^6 - 6\lambda^4 + 9\lambda^2 - 4)^5$
$\vdots$	$\vdots$
mC_6	$((\lambda - 2)(\lambda - 1)^2(\lambda + 1)^2(\lambda + 2))^m$

From Table 5, the spectrum of the m-copy cycle graph mC_6 is obtained as follows:

Theorem 3.4

The spectrum of the m-copy cycle graph mC_6 is:

$$Spec\ mC_6 = \begin{pmatrix} -2 & -1 & 1 & 2 \\ m & 2m & 2m & m \end{pmatrix}$$

Proof.

The adjacency matrix of the m-copy cycle graph mC_6 is:

$$G = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Based on matrix G, the eigenvalues and eigenvectors of this block diagonal matrix can be determined by solving $\det(G) = 0$. Thus, we obtain the following matrix:

DOI : https://dx.doi.org/10.24014/jsms.v12i1.36489

$$G-\lambda I=\begin{bmatrix}-\lambda & 1 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & -\lambda & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & -\lambda & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -\lambda & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & -\lambda & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 1 & -\lambda & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\lambda & 1 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -\lambda & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -\lambda & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -\lambda & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & -\lambda & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 1 & -\lambda\end{bmatrix}$$

Let $H=\begin{bmatrix}-\lambda & 1 & 0 & 0 & 0 & 1\\ 1 & -\lambda & 1 & 0 & 0 & 0\\ 0 & 1 & -\lambda & 1 & 0 & 0\\ 0 & 0 & 1 & -\lambda & 1 & 0\\ 0 & 0 & 0 & 1 & -\lambda & 1\\ 1 & 0 & 0 & 0 & 1 & -\lambda\end{bmatrix}$, then $G-\lambda I=\begin{pmatrix}H & 0 & \cdots & 0\\ 0 & H & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & H\end{pmatrix}$.

By Theorem 2.1:

$$det(G-\lambda I)=det(H)\times det(H)\times \dots \times det(H)$$

Next, using the cofactor expansion method as in Theorem 2.2, the determinant of matrix $H$is computed by expanding along the first row, yielding:

$$det\ (H)=(\lambda^6-6\lambda^4+9\lambda^2-4)^m$$

Since

$$det(G-\lambda I)=(\det(H))^m$$

it follows that:

$$det(G-\lambda I)=(\lambda^6-6\lambda^4+9\lambda^2-4)^m$$
$$det(G-\lambda I)=((\lambda-2)(\lambda-1)^2(\lambda+1)^2(\lambda+2))^m$$
$$det(G-\lambda I)=(\lambda-2)^m(\lambda-1)^{2m}(\lambda+1)^{2m}(\lambda+2)^m$$

Based on Definition 2.3, the eigenvalues are the roots of the characteristic polynomial. Therefore, the eigenvalues are:

$$\lambda_1=-2, \lambda_2=-1, \lambda_3=1, \lambda_4=2.$$

From these eigenvalues, and using Theorem 2.3, the multiplicities are $m(-2)=m$, $m(-1)=2m$, $m(1)=2m$, and $m(2)=m$. Hence, the spectrum can be written as:

$$Spec\ mC_6=\begin{pmatrix}-2 & -1 & 1 & 2\\ m & 2m & 2m & m\end{pmatrix}.$$

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